The discussion of the proposed Mathematical Dictionary Friday morning was in no sense in the nature of a report of the committee which was appointed some three years ago and which has made on other occasions preliminary reports. The object of this discussion was avowedly to secure a consensus of opinion among a large number of mathematicians and to gather data both pro and con, for the guidance of the committee in its future deliberations. To this end numerous individuals, including some members of the committee, presented tentative proposals concerning different phases of the project, all of which were solely on the responsibility of the individuals concerned. All of these suggestions will be of the utmost usefulness to the committee when it comes to the formulation of definite plans for carrying forward the project, such as the final determination of the scope and content of the dictionary and the distribution of the editorial responsibility. Abstracts of the papers and discussions are given below. The most important phase at this time was the determination of the desirability and feasibility of the project. On this point a vote was taken in the form of a resolution which was carried with but one dissenting voice. The chief questions of doubt raised in the discussion were (1) whether such work would interfere seriously with the output of research; (2) whether some larger project should not be undertaken, such, for example, as the founding of a journal for abstracts in the English language to fill the place formerly occupied by the *Fortschritte*. These objections were met by the various speakers as shown in the abstracts of their remarks given below. A significant fact was that in the case of several who expressed a doubt they hastened to say that they should of course want a copy of the dictionary for their own use; in the last analysis the fact that both students and experts want the dictionary is a controlling reason why it should be prepared.

The program accompanied by numbered abstracts is grouped in four parts. Professor W. B. Ford was chairman of the program committee.

Joint Session of the Association with the American Mathematical Society and the American Astronomical Society.

(1) "Mathematics and Statistics"—Retiring address of the President of the Association, Professor E. V. Huntington, Harvard University.

(2) The Work of the National Research Council with reference to Mathematics and Astronomy.—Professor E. W. Brown, Yale University.

(3) Report on the Meeting of the International Astronomical Union at Brussels.—Director Frank Schlesinger, Allegheny Observatory, University of Pittsburgh.

(4) Report on the Meeting of the International Union of Geodesy and Geophysics at Brussels.—Director L. A. Bauer, Department of Terrestrial Magnetism, Washington, D. C.

(1) The address is to be printed in the December issue of the Monthly.

(2) Professor Brown stated that when the modern scientific societies were formed, they were devoted less to forwarding the work of the various sciences

than to providing that those interested might meet one another and confer about their results. The National Research Council is to a considerable extent the result of the great war. Just as the analogous organization in Great Britain endeavored to bring the resources of English science into effective relations with the government, so in the United States certain leading scientists foresaw that it might become needful for the country to provide for certain contingencies and these men planned to provide an organization which would in case of need provide the means of marshalling the forces of science for the service of the government.

Very early the government acquired from the allies a great body of useful information on ordnance, signalling, development of aërial navigation, etc., and within but a few months after our declaration of war the work of the Research Council came into evidence. The organization is still in its original form but is now devoted to purposes of peace. One of its important activities may be characterized roughly as the encouragement of research; this may be accomplished by assistance rendered to individuals engaged in research, the removal of hindrances to such research, the provision of adequate subsidies for the apparatus of research, greater coöperation between groups of workers and between those working in different but related lines, the preparation of a national census of research, etc.

A second phase of the work of the Council is that of coöperation in activities which demand international efforts. This has been done previously in astronomy more fully than in any other subject. But there is need everywhere of a great advance. For instance, the questions of units and of notation are of great interest to astronomers and indeed to all scientists; one may cite further exchanges of scientific publications, information service, exchange of professors and of research students, collaboration on international catalogs and periodicals. Practically every one of the old organizations is dead; the International Research Council affords the agency for bringing about anew meetings of scientific men and the reading of scientific papers.

The National Research Council proposes in no sense to control research or to force coöperation as taking the place of any desired individual activity; it seeks rather to encourage research, to obtain increased assistance when required and to make a survey of the larger possibilities of research. The selection of members of the Council and subdivisions will be made mainly by the national societies of America. The subdivisions of the Council will ultimately be very numerous and will be adequate to handling these problems through sections of physicists, mathematicians, astronomers, etc.

(3) Dr. Schlesinger referred to two meetings held in the fall of 1918 at Paris and London; those in attendance were the foreign secretaries or other representatives of the national academies of the principal allied nations. Under the authority of these meetings there were to be formed international unions, as well as national sections of such unions in the individual countries. Lest such a section should assume the character of a national society in a country where

the latter is already organized the members of the American delegation to the International Astronomical Union have recommended the only permanent feature of the section shall consist of an executive committee of seven members whose chief function would be to prepare for the triennial meeting of the International Union. This is to be done by making a survey of the state of the science through the appointment of small temporary committees in specific branches of the science.

With regard to the Brussels meetings in July of this year, the American delegates were prepared to discuss technical and scientific plans, but the European delegations had considered only matters of organization. The general plan of the latter was to subdivide astronomy into a number of divisions and to set up for each of these a central bureau, each of which was to have a president, a secretary and an executive committee. The American delegation was of opinion that the Union should be made up of committees which were to deal with specific problems that require international coöperation. This latter plan was finally adopted. Thus the Carte du Ciel will henceforth be directed through a committee of the International Union, and similarly the determination of Latitude Variations and many analogous projects.

It was decided to admit neutral nations to the Union on the same basis as the allied countries; for this reason the definite organization of the Union will not begin until January 1, 1920; those of the allied and neutral nations who have signified their adherence to the Union at that date will be considered as charter members of the Union.

The number of votes for each country and the financial contributions from each were apportioned as follows:

	Votes.	Units for Financial Contributions.
Under 5 millions population	1	1
Between 5 and 10 millions	2	2
" 10 " 15 "	3	3
" 15 " 20 "	4	5
Over 20 "	5	8

The unit for financial contribution was recommended as $400 for the present. The next meeting of the Union will be held at Rome in the spring of 1920, and the next after that probably at Cambridge, England, in 1922.

(4) As finally organized, this Union, Doctor Bauer reported, consists of the following six sections: Geodesy, seismology, meteorology, terrestrial magnetism and electricity, physical oceanography, and vulcanology. The opinion was expressed generally at that meeting that in the organization of work for the various sections the endeavor should be to distribute the work among the various committees rather than centralize the investigational work at the central bureau.

In addition to the two unions described in these two addresses, there were established under the auspices of the International Research Council international unions of mathematics, of physics, of chemistry, and of scientific radio-telegraphy.

TOPICS ON THE SEPARATE ASSOCIATION PROGRAM.

(5) "Continuity in Synthetic Geometry"—PROFESSOR JOHN MATHESON, Queen's University, Kingston, Canada.

(6) "Some Aspects of Mathematics in Biology"—DR. R. B. ROBBINS, University of Michigan, by invitation.

(7) "Mathematical and Astronomical Rarities in the Library of the University of Michigan"—PROFESSOR L. C. KARPINSKI, University of Michigan.

(5) The geometry of Euclid had no generalizations but was characterized by the logical sequence of its propositions. A certain amount of generalization was attained through the so-called principle of continuity as introduced by Kepler and more fully developed by Poncelet. But a higher degree of generalization may be reached through the application of the notion of continuity as applied to ordinary mathematical functions. For this purpose definitions may be given of the limit of a point which traces out a curve and of the continuous motion of the point, of the continuous motion of a line, and of the continuous variation of quantities such as length, angle and area. Based on such definitions, there is the fundamental principle that if a constant relation holds for a number of continuous variables for all sets of simultaneous values approaching a limit, the relation holds true also at the limit. On this principle we find all through geometry groups of theorems which are forms of a common principle. Such generalizations add much both to the interest in the subject on the part of college students.

(6) The Mendelian theory of inheritance is a clear cut hypothesis of the same nature as the statement of an urn problem. It opens up for those interested in the theory of probability a variety of types of problems with the added attraction that they are suggested by the needs of another science. The purpose of this paper is to show by means of a few examples the kinds of mathematical tools needed in these problems. The problems so far considered have involved the solution of systems of recurrence relations, subject to initial conditions determined by the nature of the original matings.

Most of the mathematical work so far done along this line has been on what we call one factor problems. The two factor problems are more complicated to about the same extent that geometry of two dimensions is more complicated than that of one dimension. The problems are further complicated when we consider linked hereditary traits. The present author's contribution to the literature of this subject consists of articles in Volumes 2, 3 of *Genetics* (Princeton University Press) under the heading "Applications of mathematics to breeding problems" and a short article in *Journal of Genetics*, Vol. VII, entitled "Partial self-fertilization contrasted with brother and sister mating."

(7) The mathematical and astronomical collections of the Library of the University of Michigan compare most favorably with any collections in America to be found outside of New York City. In the Exhibit, which was arranged for the Association meeting and which was visited by many during the sessions and on Friday afternoon when Professor Karpinski personally explained the collection, were placed all the works on the history of mathematics and astronomy, a

fairly complete collection, the bibliographies, the dictionaries of mathematics and astronomy, the mathematical tables, and all mathematical and astronomical works in the library which were published before 1800.

Of particular interest is the collection of photographs (rotographs) of manuscripts from European libraries dating from the twelfth to the sixteenth centuries. These photographs include a large number of arithmetics or algorisms, as they are called after Al-Khowarizmi, in which the Hindu art of reckoning was first taught to Europeans; early Latin algebras are included, notably two Latin versions of Al-Khowarizmi's algebras and one of Abu Kamil. Recent additions to this collection include the collected works of Richard Wallingford, Abbot of St. Albans, one of the most famous English mathematicians and astronomers of the fourteenth century. Wallingford's *Quadripartitum de sinibus et chordis* has been transcribed by Professor J. D. Bond of Texas A. and M. College, a fellow in mathematics at the University of Michigan. Professor Bond made the interesting discovery that Wallingford proposed a new radius of 150, to be used in the computation of sines, instead of the radius of 60 used by Ptolemy in his table of chords. The tables exhibited include several editions of Ptolemy's tables in his *Almagest;* of particular interest is the complete five volume official edition of the Alfonsine tables, made under the direction of King Alfonse the Wise of Spain, about 1275. A number of various works of Peurbach are included, notably his *De sinibus et chordis* with tables of sines by Regiomontanus (1436–1476) to the radius 6,000,000 and 10,000,000. After the introduction of decimal fractions the tables to the radius 10,000,000 adapted themselves to the unit radius.

Of the seven incunabula exhibited three are not found elsewhere in America. These are the following:

1490, Regiomontanus, *Tabule directionum profectionum famosissimi viri Magistri Ioannis Germani de Regiomonte.* Augsburg, Ratdolt, 2. Jan, 1490; written 1467.

1495, Bradwardine, *Geometria Thome brauardini cum tractatulo de quadratura circuli bene reuisa a Petro sanchez ciruelo:* operaque Guidonis mercatoris diligentissime impressa Parisiis in campo gaillardi Anno domini 1495, die 20 maij.

1499, Sacrobosco, *Opusculum Ioannis de sacro busto spericum cum notabili commento. atque figuris textum declarantibus vtilissimis.* Leipzig, Wolfgang Stöckel, 1499.

By the courtesy of the University Library pamphlets with a brief description by Professor Karpinski of the University collections in mathematical lines were available for the visitors.

Discussion of the Proposed Mathematical Dictionary.

(8) Its General Desirability and Feasibility. Professor E. R. Hedrick, University of Missouri.

(9) Its Possible Scope and Content. Professor G. A. Miller, University of Illinois.

(10) Possible Distribution of Editorial Responsibility. PROFESSOR R. C. ARCHIBALD, Brown University.

(11) General discussion, in person or by letter, by PROFESSORS D. E. SMITH, Columbia University, L. E. DICKSON, University of Chicago, L. P. EISENHART, Princeton University, and E. W. BROWN, Yale University, and others.

(8) Professor Hedrick rehearsed briefly the organization, previous reports, and present organization of the Committee on a Mathematical Dictionary. The general need for such a work and the activity of the Committee both in outlining the work and in seeking adequate financial support were mentioned. Recently, through the activity of Professor D. E. Smith, definite proposals looking toward financial aid have been made to organizations disposing of funds.

The relation of this work to the larger field of activity of the Committee on the Apparatus of Research of the American Association of University Professors was mentioned. This other committee has already initiated important work for which financial support has been secured; and it stands ready to aid the dictionary project directly, if funds can be secured for such general purposes. The committee is also interested in such works of reference as those formerly published in Germany in which abstracts of articles appeared. It is understood that this phase, so far as mathematics is concerned, is already under consideration by the American Mathematical Society.

The view was expressed that all such work, throughout the field called the Apparatus of Research, will devolve more and more upon America. This dictionary project, on account of its relative simplicity and on account of its natural terminability, offers a safe means of initiating such work in this country, to train men toward such work and to determine its feasibility here.

Another sort of relation concerns possible interference with scientific research on the part of men engaged in it. To avoid this, a very detailed plan, involving the extensive use of clerks to do much clerical work, was proposed.. While this plan cannot be repeated in detail here, it may be said that its purpose would be to relieve research men entirely of needless clerical work.

The conclusion drawn was that the entire project is desirable and entirely feasible, and that without undue interference with research activities, provided always that funds can be secured. Finally, to uphold the hands of those engaged in the attempts to secure such funds, it was urged that unequivocal resolutions be passed in support, not only of this dictionary project, but also, in a much wider field, of all those projects under consideration by the Committee on the Apparatus of Research.

(9) In Professor Miller's judgment all terms which are usually sufficiently defined in the ordinary dictionaries and encyclopedias should be excluded from this dictionary. Personal names should not be included among the terms to be explained in view of the fact that the number of these names is so large; this can be more adequately provided in a separate work, which indeed this Association might well look forward to undertaking at some future time, especially so far as it relates to American mathematicians.

If, to be concrete, we speak of group theory, it would be unwise to start the student with the original memoirs of Cauchy; these are for the most part difficult to follow and have been put into better form in modern textbooks on group theory. The large mathematical encyclopedias are invaluable to those who aim to use original sources to the best advantage, and the speaker believes that a mathematical dictionary, properly constructed, will be of equally great value to those who wish to make the most effective use of original material.

All technical mathematical terms found in the reputable literature should be included, the varied needs of the first year graduate student being especially considered. This would probably make a work of 1,500,000 words in three or four volumes comprising the definition of, say, ten thousand terms.

Professor Miller gave a somewhat detailed plan whereby 100 to 200 terms might be treated in expository articles by experts in the several fields, and the definitions of the remaining terms could be prepared first by assistants and later by associate editors in such a manner as to relieve research men of the bulk of the demands which might otherwise make serious inroads upon their research.

(10) Professor Archibald stated that of several important undertakings that demand intense coöperative effort on the part of mathematicians it is the dictionary project alone which we, as a nation, are equipped to carry through unaided. The great need of a new mathematical dictionary was expressed twenty-five years ago, especially by the French,[1] existing dictionaries being more than fifty years old and Müller's French-German and German-French mathematical vocabulary not performing adequately the function of a dictionary. In comment upon Professor G. A. Miller's article published in this MONTHLY[2] to illustrate the nature of a possible general article for the proposed dictionary, Professor Archibald called attention to a criticism of this article by P. E. B. Jourdain[3] and expressed his judgment that the order and extent of inclusion of foreign languages should be French, German, Italian, Swedish, Dutch, and possibly Russian or Spanish. He would greatly increase the value of this collection of terms by arranging those in each language alphabetically with the English equivalents opposite; this could be done for 12,000 terms in each of five foreign languages in an extra volume of 500 pages.

He then presents a plan whereby not two editors, as already proposed, but four editors should devote to the work all their time for two years, these four to select a consulting board of ten whose chief duty would be to read, and to suggest possible revisions of, all articles written. Professor Archibald thus indicates his conviction that the labor of drawing up first drafts of thousands of articles is sure to devolve upon the editors; herein his plan differs materially from some of the others. A suggestion is made of the possibility of requesting some scholars from the British Empire to share in the undertaking, although the bulk of preparation would still fall to Americans. Such a plan as here set forth, it is estimated, would require a budget of sixty-three thousand dollars.

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[1] AMER. MATH. MO., Vol. I, p. 368.
Paul Tannery, *Bulletin des sciences mathématiques*, 1899, vol. 22, pp. 165–167; see also vol. 24, pp. 25–27.

[2] November, 1918, pp. 383–387.

[3] *Science Progress*, April, 1919.

(11) Various estimates of the financial needs in the preparation of the Dictionary were proposed, ranging from $30,000 by Professor Smith to $63,000 by Professor Archibald, depending on the number of paid experts who might be employed on salaries and on the length of time that might be involved. But it was the consensus of opinion that it would surely prove to be an expensive undertaking and that its high character should in no way be compromised by stinting the expenditure of funds in its preparation. Moreover it was pointed out that a project whose far reaching importance demands large financial backing will be much more likely than one of smaller nature to appeal favorably to those who are in a position to provide such funds. To quote Professor Smith, "It is evident that a dictionary could be prepared for less money. If it cost half as much, I should think it would be worth about one tenth as much. If we go into the matter, I feel that we should produce a work that will be a standard in Europe as well as America for many years to come,—one which will be an honor to American scholarship and to American book manufacturing."

Professor Dickson in a written communication said: "As to the proposed Mathematical Dictionary, I have been convinced by reflection and conversations with others that it would prove to be a valuable aid to mathematics, partly to experts when reading outside their specialties, but mainly to amateurs, to students, and to persons early in their mathematical careers. I believe many overlook the great extent of technical mathematical language in the literature and the fact that difficulties of students are very largely due to their failure to understand this technical language. Here is where the main service of the dictionary would come in. The fact that our technical words are in English causes us to focus attention on them less than would be the case if they were in Latin as in botany and medicine, with the result that we do not really learn thoroughly most of the technical shades of meaning.

"A mathematical dictionary which would collect by aid of experts all the *technical* uses of words and phrases in mathematics would be of great value to research men and of still greater value to all students of mathematics.

"No project is ever endorsed unanimously. The fact that the Dictionary is so generally approved seems to me a proof that the idea is sound. Nor do I fear any dissipation of mathematical energy in the preparation of the manuscript. There are enough well equipped men not putting their time mainly on research to do the work without lessening the research output. As there is no other big mathematical project afoot, it seems to me that there is nothing to lose in this Dictionary project, while I am one of those who strongly believe that there is considerable serious value to gain in general and that the pedagogical value is undisputed."

Professor Eisenhart suggested that some valuable aid in the compilation of definitions might be secured from experts in dictionary making. It is important that Americans do much more than merely prepare a limited number of papers each year; judging by these only a small number of our mathematicians are doing real research work. Mathematicians should be so organized that they

might be stimulated to do further work and to acquaint themselves with particular fields of study; for example, in the making of this dictionary many such might write up the definitions, the experts in various fields criticizing and selecting these.

Professor Shaw expressed his belief that we have a large body of men capable of doing this work. His belief is based on an experience with a successful plan carried out at the University of Illinois whereby the staff coöperate in keeping the run of mathematical journals, each one reporting on the matter found in some particular journal. Incidental to such reading as this, one often needs a mathematical dictionary when reading in a field to one side of his own special field.

Professor Ford said that the dictionary makers must exercise care and see to it that the definitions are of the sort that the users need,—simple, pithy definitions with clear examples. He showed several dictionaries, among these the American Medical Dictionary which in four or five lines for each word gives its derivation, and an English-German-French Technological Dictionary.

Professor Huntington instanced the attitude which the non-mathematician often holds regarding mathematics, quoting a dismayed reviewer of a book on statistics, who said that the technical terms in whose scientific exactness mathematicians pride themselves are forbidding and impossible. A mathematical dictionary would help such outsiders.

Professor Richardson raised doubt whether the dictionary project is the task for American mathematicians for the immediate future. He stated that we should attempt something at once to replace the *Fortschritte*, and that this would not be possible if we were to attempt the dictionary; a translation of Weber-Wellstein or of the new edition of Pascal's *Repertorium* would supply the need for the longer article which is the more important aspect of the project.

Professor Brown stated that he would undoubtedly order three copies of the dictionary for himself and the libraries, but he finds (1) a scientific difficulty in defining many words which have both a mathematical and a popular meaning, e.g., addition, and (2) business difficulties, particularly as to the relative importance of this project and the plan to institute a proposed journal to replace the *Fortschritte* and an Anglo-American or an English encyclopedia.

In answer to the last two speeches, Professor Lunn told of the scheme for abstracts now being used by the *Physical Review*, whereby the author makes his own abstract and this is revised by the editor, such a plan serving to save effort in making reviews of the type of those in the *Fortschritte* and to avoid the foregoing criticism of the loss to research; and Professor Hedrick stated his sympathy with the *Fortschritte* plan but desired to see the dictionary scheme completed which now seems immediately feasible.

Mr. Webb expressed his hope for an early publication of a smaller edition of the proposed dictionary.

At the completion of the discussion, as already stated, the following resolution was adopted:

Be it resolved by the Mathematical Association of America that the prepara-

tion and publication of a dictionary of mathematical terms is not only most desirable, but also entirely feasible, provided that financial aid for editorial guidance and clerical assistance can be secured; and

Be it resolved, that the Association views with great interest and hope the efforts of a national committee on the Apparatus of Scholarship, for the promotion of publication of works of a reference nature, including works for abstracting and reporting progress in the various sciences, for which there is and promises to be the most pressing need in all scientific, scholarly, and educational work.

SYMPOSIUM UPON THE PRESENT DAY RELATIONS AND TENDENCIES BETWEEN THE HIGH SCHOOLS AND THE COLLEGES AS REGARDS MATHEMATICS.

(12) From the standpoint of the High School as a fitting school for College. PROFESSOR H. E. WEBB, Central High School, Newark, N. J.

(13) From the standpoint of the College as a training school for High School teachers. DR. E. B. LYTLE, University of Illinois.

(14) From the standpoint of coördinated institutions in a general system of education. PROFESSOR B. F. YANNEY, College of Wooster.

(15) The Work of the National Committee on Mathematical Requirements. PROFESSOR J. W. YOUNG, Dartmouth College.

(12) Ten years ago much ado was made of the inutility of the traditional canon of secondary mathematics. It was fondly hoped that problems of real import to the external world would bring to mathematics an added interest. The undesirable results are too well known to bear recounting. Yet these were not altogether disastrous. We are, for example, in a fair way to eradicate the endless chain of competition between the examiner who invents new and unusual algebraic forms for solution and the tireless private tutor who carefully codifies all those forms for future recognition by his pupils. The speaker believes that one of the present tendencies in secondary teaching with a view to college preparation is toward a greater emphasis upon clarity and accuracy in the formulation of fundamental principles of algebra and geometry as opposed to a constantly advancing standard of intricacy in illustration.

Again the insistence upon real problems has convinced those students who have a native interest in abstract mathematics that this science is primarily intended for use rather than for amusement.

A consideration of the curriculum of the secondary schools for the past generation leads the speaker to the conclusion that one and only one foreign language for entrance credit shall be sufficient save for the possible exception of one or two years of Latin preceding a romance language. This would allow time for a thorough mathematical preparation and for adequate drill in English. A detailed plan for four years of mathematics on this basis was then given with due attention to continuity in these courses.

On the other side of the question the requirements for college entrance should include a certain degree of maturity of mind, and possibly of body, and such specific demands in particular subjects as are prerequisite for further study.

Thus the responsibility for secondary training is placed exactly where it belongs, —upon the authorities in the secondary field. The earnest desire was expressed that this Association might put itself upon record as against the utter looseness of election of secondary studies. Finally, in view of the extension of high school instruction into the domain of college work under competent instruction, Professor Webb pled for greater liberality in admission to advanced standing and for a standardization of freshman and sophomore mathematics.

(13) Besides calling attention to the importance of high-school teacher training, and giving some facts and data concerning the present status of teacher training in our colleges and universities, Dr. Lytle developed for discussion suggestions that American standards of secondary mathematics teacher training may be raised (1) by emphasizing the desirability of the greatest possible scholarship in mathematics, (2) by developing stronger courses in theory of equations, advanced geometry, fundamental concepts and history, (3) by emphasizing in all our courses perspectives and large ideas, (4) by developing the use of libraries, knowledge of the best literature and ability to prepare and forcefully present papers on mathematical topics, (5) by making one member of the staff responsible for investigation in the field of teacher training, (6) by encouraging professional training specially in adolescent psychology, (7) by taking constructive and not destructive attitudes toward scholarly investigation in education, (8) by demanding that supervisors of mathematics teacher training work have the degree of Ph.D. or the equivalent in mathematics as well as special training in education, (9) by studying the Wisconsin "directed teaching" plan with the purpose of modifying it to apply to college classes, and (10) by writing and translating books or articles which appeal to the present interests and attainments of high-school teachers.

(14) Professor Yanney first gave a historic review, beginning with the Colonial period and ending with the world war. Starting with no mathematics on the boundary line between the two units of our educational system, the varying changes were noted, culminating with one unit of algebra and one unit of plane geometry as fairly well established on the high school side of this boundary line.

Then followed a reference to organizations, institutions and individuals interested in various problems of relations between the two educational units, together with a description of the activities now engaging the attention of these organizations and institutions. These include both entrance problems and problems connected with the preparation of high school teachers in the colleges.

The present relations were characterized as cordial, coöperative, and constructive, with a number of details yet to be worked out before standard practices now well started become generally prevalent. There are standard colleges now articulating their collegiate courses in mathematics with preparatory courses comprising two, or three, or four units of mathematics. Two distinct current tendencies were mentioned. (1) One was the breaking up into smaller units, along the border line, of secondary school work and college work,—on the one side the junior and senior high schools, and on the other the junior college.

This was regarded as favorable to a still better articulation of mathematics between high school and college. (2) Another tendency referred to was that of the breaking up of the traditional compartments in mathematics, a movement in evidence on both sides of the boundary line already referred to. A third strong tendency, affecting both high school and college mathematics, was pointed out to be that of reappraising educational values as a result of experiences in the recent upheaval, educational aspects having become more or less international. This would eventuate, it was predicted, in increased interest in the study of mathematics in both the high school and the college.

(15) Professor Young reported that it was early found by the National Committee on Mathematical Requirements that it would be impossible to get their plans under way without larger funds and that finally, last spring, the General Education Board interested itself in this work. In the endeavor to reorganize and to strengthen the secondary school curriculum, the committee now purposes to obtain the names and addresses of teachers of high schools and junior high schools, to interest them in the activities of the committee, to make a study of the educational systems in the various states, to furnish information and speakers for the various organizations interested. It is expected that contact will be made with perhaps fifty or sixty such organizations through a representative officer of each.

A bulletin is projected which shall serve as a means of intercommunication between the committee and the great body of teachers. The committee looks to the members of the Association and others over the country for all possible assistance and suggestions as to the furtherance of its purposes. A fuller report in these pages may be expected at an early date.

MEETING OF THE COUNCIL OF THE ASSOCIATION.

Ten members of the Council attended the two sessions held on Thursday and Friday.

The following twenty-two persons and one institution, on applications duly certified, were elected to membership:

To individual membership.

C. R. ADAMS, A.B. (Brown). Instr., Brown University, Providence, R. I.

MAELYNETTE ALDRICH, Ph.D. (Yale). Prof., Martha Washington College, Abingdon, Va.

STUART BALLANTINE. Radio Research Engineer, U. S. Navy Dept., Philadelphia, Pa.

SARAH BEALL. Computer, Coast and Geodetic Survey, Washington, D. C.

FLORENCE L. BLACK, A.B. (Kansas). Instr., Univ. of Kansas, Lawrence, Kans.

T. M. BLAKSLEE, Ph.D. (Yale). Retired, Des Moines College, now of Ames, Ia.

J. G. COFFIN, Ph.D. (Clark). Asso. prof., Coll. of City of New York, on leave of absence; Asst. dir. of research, Curtiss Engg. Corp., Garden City, L. I.

J. M. FOSTER, A.M. (Rochester). Supt. of Schools, Corning, N. Y.

W. W. HART, A.M. (Wisconsin). Asst. prof., Univ. of Wisconsin, Madison, Wis.

C. A. ISAACS, A.M. (Columbia). Head of dept. of math., State Coll. of Washington, Pullman, Wash.

M. F. JOHNSON, C.E. (Mich. Agric. Coll.), M.S. (Michigan). Instr., Univ. of Michigan, Ann Arbor, Mich.

E. C. KIEFER, M.S. (Michigan). Asst. prof., Iowa State Coll., Ames, Ia.

G. R. LIVINGSTON, B.S. (California). Head of dept. of math., Junior Coll., Santa Barbara, Calif.

ANNA MARM, A.M. (Kansas). Instr., Univ. of Kansas, Lawrence, Kans.

E. W. MARTIN, S.B. (Chicago). Prof., Wilmington Coll., Wilmington, Ohio.

ANNIE MCK. PEGRAM, A.M. (Trinity Coll., N.C.; Columbia). Prof., Coll. for Women, Greensboro, N. C.

J. R. SAGE, M.S. (Rose Polytechnic). Asst. prof., Iowa State Coll., Ames, Ia.

Mrs. CATHERINE SELVES, B.S. (Kirksville State Normal). Instr., La Grange Coll., La Grange, Mo.

H. B. SMITH, A.B. (Kenyon). Lt., U. S. Coast Artillery, Fort Caswell, N. C.

C. N. STOKES, A.M. (Illinois). Head of dept. of math., McKendree Coll., Lebanon, Ill.

R. R. TILESTON, A.M. (Dartmouth). Prof. of physics, Colorado Coll., Colorado Springs, Colo.

ARTHUR WALTER, A.M. (Stanford). Asst. and graduate student, Stanford Univ., Salinas, Calif.

To institutional membership.

WEST CHINA UNION UNIVERSITY, Chengtu Sze Chwan, W. China.

It was voted to be the sense of the Council that the annual meeting of the Association be held at New York City provided that like action is taken by the Society.

It was voted that a committee, with Professor Eisenhart as chairman, be appointed by the president to investigate the question of junior branches or of associate membership in the Association.

The Council considered informally the financial statement of the treasurer and the question of the extension of the arrangement between the Association and the *Annals of Mathematics* beyond the original three-year term.

It was voted to increase to four dollars the subscription price of the MONTHLY to non-members, beginning with January, 1920, in view of the notification from the printers of an increased charge for printing and in view of the reasonableness of distinguishing between the general subscription price and a reduced price to members of the Association.

It was voted that the Council request the president and the chairman of the Committee on Dictionary to present the matter of the dictionary project to the Council of the Society and to request their endorsement of the project.

W. D. CAIRNS, *Secretary-Treasurer.*

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

In the first discussion this month Professor Moritz shows a method of deriving the area bounded by a parabola and certain straight lines by principles of elementary character such as are used in obtaining the area of a circle in plane geometry. The method applies with equal ease to the ordinary quadratic parabola and to parabolas of higher order. It is not likely that this proof will be entirely intelligible to all the students of an ordinary high-school class; but the same statement holds in connection with the area of the circle. It is worthy of note that the validity of Professor Moritz's proof depends on the possibility of saying that the quantity which he calls F^2 remains finite for all choices of r and n; this is easy to see on writing out the value of F^2.

Mr. Haldeman shows how to solve a cubic equation graphically by means of ruler, compass, and an appropriately chosen equilateral hyperbola, and applies the result to the graphical construction of the side of a regular heptagon inscriptible in a given circle. Since equilateral hyperbolas differ only by translations, rotations, and similarity transformations, it seems possible that the construction could be performed by the use of ruler, compass, and a given fixed equilateral hyperbola.

Professor Schmiedel indicates a method for obtaining the sum of a definite number of terms of certain types of series, and gives some interesting interpretations of the results. All the series considered are of frequent occurrence in connection with Fourier series and other allied developments.

The fourth discussion is a short note by Mr. M. W. Jacobs on the reason for the occasional success of a false rule for finding the hypotenuse of a right triangle in terms of the two sides. The case actually considered, even in the extended form of the corollary, is so simple as to be almost obvious. If we ask when it is possible for the hypotenuse to be represented linearly with rational coefficients (neither being required to be unity) in terms of the perpendicular sides, we have a slightly more complicated case, which leads to a familiar type of Diophantine equation. May we have the discussion of this case also?

I. ON THE QUADRATURE OF THE PARABOLA.

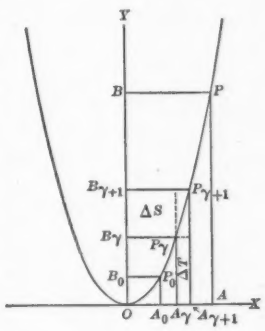
By R. E. MORITZ, University of Washington.

While most books on analytical geometry derive the formula for the area of an ellipse from its relation to the area of the circumscribed circle, the attempt is but seldom made to derive the formula for the area of a segment of the parabola.

The reason for this must be sought in the fact that the classic proofs of the formula in question require principles which are foreign to the methods of analytical geometry. Thus the famous proof by Archimedes involves, besides ten or eleven preliminary propositions, the sum of the infinite series $\Sigma(1/n^2)$. On the other hand, the method of infinitesimals (indivisibles) first applied to the quadrature of the parabola by Cavalieri and later extended by Wallis to the quadrature of the curve $y = x^p$, involves the limit of the sum $(1^p + 2^p + 3^p + \cdots + n^p)/(n^{p+1})$ as n approaches infinity.

The following proof is probably not new though the writer is not aware that it is recorded in any of the sources available to him. It is certainly far more simple and direct than any of the classic proofs. In fact it involves no principles except such as are familiar to every student of elementary algebra and geometry. Even when extended to the quadrature of the general parabola, $y = px^m$, it comes well within the range of the average high-school student.

Quadrature of the Parabola $y = px^m$. Let $P_0(x_0, y_0)$, $P(x, y)$, represent any two points on the parabola which lie on the same side of the vertex O, and let it be required to determine the area included between the curve, either axis, and the perpendiculars from these points on the y-axis.



From P_0 and P draw the perpendiculars P_0A_0 and PA to the x-axis. Divide A_0A into n equal parts and call the length of each part h. At the points of division, A_r, erect perpendiculars and through the points P_r, in which these perpendiculars meet the curve, draw lines parallel to the x-axis intersecting the y-axis in the points B_r. Denote the area B_0P_0PB by S, the area APP_0A_0 by T; the area of the rectangle P_rB_{r+1} by ΔS, and the area of the rectangle P_rA_{r+1} by ΔT. Then

$$\Delta S = x_r(y_{r+1} - y_r) = px_r(x_{r+1}^m - x_r^m),$$

$$\Delta T = y_r(x_{r+1} - x_r) = px_r^m(x_{r+1} - x_r),$$

$$\frac{\Delta S}{\Delta T} = \frac{x_{r+1}^m - x_r^m}{x_r^{m-1}(x_{r+1} - x_r)} = \frac{x_r^{m-1} + x_r^{m-2}x_{r+1} + x_r^{m-3}x_{r+1}^2 + \cdots + x_{r+1}^{m-1}}{x_r^{m-1}}$$

$$= 1 + \left[1 + \frac{x_{r+1} - x_r}{x_r}\right] + \left[1 + \frac{x_{r+1}^2 - x_r^2}{x_r^2}\right] + \cdots + \left[1 + \frac{x_{r+1}^{m-1} - x_r^{m-1}}{x_r^{m-1}}\right]$$

$= m + hF^2$, since each of the binomials is positive and divisible by $x_{r+1} - x_r$ which is equal to h.

Now as n is increased indefinitely, h approaches 0, hence $\Delta S/\Delta T$ approaches m, the sum of all the ΔS's approaches S, the sum of all the ΔT's approaches T, hence in the limit

$$S = m \cdot T.$$

Adding first mS and then T to both sides of the equation we find

$$(1+m)S = m(S+T), \qquad S+T = (1+m)T,$$

hence

$$S = \frac{m}{1+m}(S+T), \qquad T = \frac{1}{1+m}(S+T).$$

Now obviously $S+T = xy - x_0y_0 = p(x^{m+1} - x_0^{m+1})$, so that we may write

$$S = \frac{mp}{1+m}(x^{m+1} - x_0^{m+1}), \qquad T = \frac{p}{1+m}(x^{m+1} - x_0^{m+1}).$$

If the foregoing proof seems to lack rigor, the objection may be easily removed as follows. We have shown that $\Delta S/\Delta T = m + hF^2$ and $\Delta T = phx_r^m$, hence

$$\Delta S = m\cdot\Delta T + hF^2\cdot\Delta T = m\cdot\Delta T + ph^2F^2x_r^m,$$

$$\Sigma\Delta S = m\cdot\Sigma\Delta T + ph^2\Sigma(F^2x_r^m).$$

In the limit $\Sigma\Delta S = S$, $\Sigma\Delta T = T$, hence in the limit

$$S = mT + \lim\,[ph^2\Sigma(F^2x_r^m)],$$

and it remains to show that the second term on the right is 0.

Let G^2 represent the greatest of the n values of $F^2x_r^m$, then $\Sigma(F^2x_r^m) < n\cdot G^2$, and $ph^2\Sigma(F^2x_r^m) < ph^2nG^2 = ph(x - x_0)G^2$, since $hn = x - x_0$, and therefore

$$\lim\,[ph^2\Sigma(F^2x_r^m)] = \lim\,[ph(x-x_0)G^2] = p(x-x_0)G^2\cdot\lim\,h = 0.$$

In the case of the common parabola, $m = 2$, $F^2 = 1/x_r$, and $G^2 = x$. In that case, too, the foregoing proof holds for any segment of the parabola. For if the line through the middle point of the chord and parallel to the axis of the parabola be chosen for the y-axis, and the tangent parallel to the chord for the x-axis, the equation of the parabola remains unchanged, and areas of the parallelograms corresponding to ΔS and ΔT are equal to $\Delta S\cdot\sin\theta$ and $\Delta T\cdot\sin\theta$ respectively, θ being the angle between the coördinate axes. The ratio of these areas remains therefore unchanged and the conclusion regarding the ratio of S to T remains valid.

II. Geometrical Construction of the Roots of a Cubic, and Inscription of a Regular Heptagon in a Circle.

By C. B. Haldeman, Ross, Butler County, Ohio.

1. The equation

$$q(x^2 - y^2) + 2x\sqrt{-(r^2+q^3)} + 2ry = 0 \qquad (1)$$

represents a real equilateral hyperbola when q^3 is negative and numerically greater than r^2, and

$$x^2 + y^2 = -4q \qquad (2)$$

is an equation of a circle, which will be real when q is negative. Eliminating x from these equations, we have

$$(y^3 + 3qy + 2r)(qy - 2r) = 0.$$

The first of these factors is a form to which the general cubic equation may be reduced. From this it appears that the three real roots of a cubic equation, reduced to the above form, may be represented by the ordinates of the intersections of an equilateral hyperbola and a circle. The three intersections, whose ordinates are the three real roots of this equation, are the vertices of an equilateral triangle; because the roots are

$$y = -2\sqrt{-q}\,\sin\tfrac{1}{3}\sin^{-1}\frac{r}{q\sqrt{-q}},$$

$$y = 2\sqrt{-q}\,\sin\tfrac{1}{3}\left(\pi + \sin^{-1}\frac{r}{q\sqrt{-q}}\right),$$

$$y = -2\sqrt{-q}\,\sin\tfrac{1}{3}\left(\pi - \sin^{-1}\frac{r}{q\sqrt{-q}}\right);$$

since $y = -2S\sqrt{-q}$ substituted in $y^3 + 3qy + 2r = 0$ gives $3S - 4S^3 = \dfrac{r}{q\sqrt{-q}}$; and comparing this with

$$3\sin A - 4\sin^3 A = \sin 3A,$$

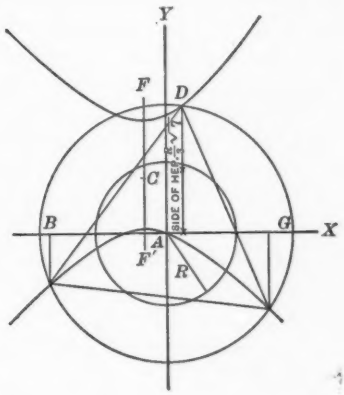
which is well known, we obtain the values of y.

2. The equation $z^3 + Rz^2\sqrt{7} - R^3\sqrt{7} = 0$ gives the value of z which represents the side of a regular heptagon inscribed in a circle whose radius is R. The transformation

$$z = y - \frac{R\sqrt{7}}{3}$$

will reduce this equation to

$$y^3 - \frac{7R^2y}{3} - \frac{13R^3\sqrt{7}}{27} = 0.$$



Comparing this with the equation

$$y^3 + 3qy + 2r = 0,$$

we have $q = -7R^2/9$, $r = -13R^3\sqrt{7}/54$; also we have the radius of (2) equal to $2R\sqrt{7}/3$.

With A as a center and $2R\sqrt{7}/3$ as a radius describe a circle BGD and with A as the origin of coördinates, after substituting the values of q and r just found in (1), draw the equilateral hyperbola

$$x^2 - y^2 + \frac{R\sqrt{21}}{7}x + \frac{13R\sqrt{7}}{21}y = 0$$

with the diameter BG prolonged as the axis of X and AY as the axis of Y. Then will the ordinate of intersection at D minus $R\sqrt{7}/3$ be the side of a regular heptagon inscribed in a circle whose radius is R; that is,

$$z = \frac{R\sqrt{7}}{3}[2 \sin \tfrac{1}{3}(\pi + \sin^{-1} \tfrac{1}{14}\sqrt{3}) - 1]$$

is the side of the regular heptagon required.

III. On the Summation of Certain Series.

By O. Schmiedel, Parsons College.

The summation of a finite number of terms of series such as

$$1 + 3x + 6x^2 + 10x^3 + \cdots,$$

$$1 + \cos\theta + \cos 2\theta + \cos 3\theta + \cdots,$$

$$1 + x\cos\theta + x^2\cos 2\theta + x^3\cos 3\theta + \cdots,$$

$$1 + 3x\cos\theta + 6x^2\cos 2\theta + 10x^3\cos 3\theta + \cdots,$$

and the evaluation of the complete series is a problem of frequent occurrence, and is easily solved by the use of De Moivre's Theorem;[1] but greater insight to the student into their nature and mutual relation is had by direct application of the more common processes of analysis.

It is proposed to produce a formula sufficiently comprehensive to embrace all series of the types suggested, including as particular cases the binomial formula for negative integral exponents and the theorem that the difference of the same powers is divisible by the difference of the first.

Let the second of the series above be deduced first, for illustration.

Assuming the identity

$$2\sin\theta/2\cos t\theta = \sin(t + 1/2)\theta - \sin(t - 1/2)\theta,$$

assigning to t values from 0 to n, and adding the results, gives directly the expression for the sum of the series of cosines, namely:

$$(1) \qquad \sum_{t=0}^{t=n} \cos t\theta = 1/2 + 1/2\frac{\sin(n + 1/2)\theta}{\sin\theta/2} = \frac{1}{\sin\theta/2}\sin\frac{n+1}{2}\theta\cos\frac{n}{2}\theta.$$

In like manner is found the sum of sines:

$$(2) \qquad \sum_{t=0}^{t=n} \sin t\theta = \frac{1}{\sin\theta/2}\sin\frac{n+1}{2}\theta\sin\frac{n}{2}\theta;$$

[1] Chrystal, *Algebra*, Vol. II, 2 ed., p. 273.

and the quotient of (2) by (1) gives the interesting relation

$$\frac{\sum_{t=0}^{t=n} \sin t\theta}{\sum_{t=0}^{t=n} \cos t\theta} = \tan \frac{n}{2}\theta, \tag{3}$$

an extension of the familiar formula $\tan \theta/2 = \sin \theta/(1 + \cos \theta)$.

Formula (1) may also be written thus:

$$\sum_{t=0}^{t=n} \cos t\theta = 1/2 + \frac{\cos n\theta - \cos(n+1)\theta}{2(1 - \cos \theta)}, \tag{4}$$

$$-\sum_{t=-n+1}^{t=-1} \cos t\theta = 1/2 + \frac{\cos n\theta - \cos (n-1)\theta}{2(1 - \cos \theta)}. \tag{5}$$

Analogously the summation of the other series is effected, as:

$$\sum_{t=0}^{t=n} x^t \cos t\theta = \frac{1 - x\cos\theta - x^{n+1}\cos(n+1)\theta + x^{n+2}\cos n\theta}{1 - 2x\cos\theta + x^2}, \tag{6}$$

$$-\sum_{t=-n+1}^{t=-1} x^t \cos t\theta = \frac{1 - x\cos\theta - x^{-n+1}\cos(n-1)\theta + x^{-n+2}\cos n\theta}{1 - 2x\cos\theta + x^2}, \tag{7}$$

one or the other of which is convergent for a given $x \neq \pm 1$, as n becomes infinite. Using the following definition and identities:

$$c_{a,\,b} = \frac{a(a-1)\cdots(a+1-b)}{1.\ 2.\ \cdots b};$$

$$c_{a,\,b} + c_{a,\,b+1} = c_{a+1,\,b+1}, \qquad c_{a,\,b} = \sum_{t=0}^{t=n} (-1)^t c_{n,\,t} c_{a+n-t,\,b-t};$$

$$2\cos u\cos v = \cos(u+v) + \cos(u-v),$$

$$\cos r\theta(1 - 2x\cos\theta + x^2)^m = \sum_{t_1=0}^{t_1=2m} \sum_{t=0}^{t=t_1} c_{m,\,t_1-t} c_{m,\,t} (-x)^{t_1} \cos(r + 2t - t_1)\theta;$$

the sums to n terms are found successively by the same process for all the series under this head, as:

$$(1 - 2x\cos\theta + x^2)^2 \sum_{t=0}^{t=n} c_{1+t,\,t}\, x^t \cos t\theta = \sum_{t=0}^{t=2} (-1)^t c_{2,\,t} x^t \cos t\theta$$
$$+ \sum_{t_2=0}^{t_2=3} \sum_{t_1=0}^{t_1=1} \sum_{t=0}^{t=1} (-1)^{t_2+t} c_{2,\,t_2-t_1} c_{2,\,2+t_1-t} c_{n+t,\,1} x^{n+1+t_2} \cos(n+1+2t_1-t_2)\theta, \tag{8}$$

$$(1 - 2x\cos\theta + x^2)^3 \sum_{t=0}^{t=n} c_{2+t,\,2} x^t \cos t\theta = \sum_{t=0}^{t=3} (-1)^t c_{3,\,t} x^t \cos t\theta$$
$$- \sum_{t_2=0}^{t_2=5} \sum_{t_1=0}^{t_1=2} \sum_{t=0}^{t=2} (-1)^{t_2+t} c_{3,\,t_2-t_1} c_{3,\,3+t_1-t} c_{n+t,\,2} x^{n+1+t_2} \cos(n+1+2t_1-t_2)\theta, \tag{9}$$

[1] *Ibid.*, p. 327 ff.

etc., and finally the general formula sought:

$$(1 - 2x\cos\theta + x^2)^m \sum_{t=0}^{t=n} c_{m-1+t,\,m-1}x^t \cos t\theta = \sum_{t=0}^{t=m} (-1)^t c_{m,\,t}x^t \cos t\theta$$

$$(10) \qquad + (-1)^m \sum_{t_2=0}^{t_2=2m-1} \sum_{t_1=0}^{t_1=m-1} \sum_{t=0}^{t=m-1} (-1)^{t_2-t} c_{m,\,t_2-t_1} c_{m,\,m+t_1-t} c_{n+t,\,m-1}$$

$$\times x^{n+1+t_2}\cos(n + 1 + 2t_1 - t_2)\theta,$$

$$-(1 - 2x\cos\theta + x^2)^m \sum_{t=-n+1}^{t=-1} c_{m-1+t,\,m-1}x^t \cos t\theta = \sum_{t=0}^{t=m} (-1)^t c_{m,\,t}x^t \cos t\theta$$

$$(11) \qquad + (-1)^m \sum_{t_2=0}^{t_2=2m-1} \sum_{t_1=0}^{t_1=m-1} \sum_{t=0}^{t=m-1} (-1)^{t_2-t} c_{m,\,t_2-t_1} c_{m,\,m+t_1-t} c_{-n+t,\,m-1}$$

$$\times x^{-n+1+t_2}\cos(-n + 1 + 2t_1 - t_2)\theta,$$

where n and m are positive.

Of the two expressions in the second member of (10), which constitute the sum of the series in the first, one is free from n, hence constant as to n; the other contains n, and hence changes with it. If for increasing n the second becomes less and less, with zero as limit, the series is convergent.

A rigorous proof of (10) by the $(n + 1)$-rule, though somewhat lengthy, is not difficult to give.

With θ made zero, (10) properly reduced, assumes the special form:

$$(12) \qquad (1 - x)^m \sum_{t=0}^{t=n} c_{m-1+t,\,m-1}x^t = 1 - \sum_{t=0}^{t=m-1} c_{n+t,\,t}(1 - x)^t x^{n+1},$$

which can be viewed in three different aspects.

Written in the form:

$$\sum_{t=0}^{t=n} c_{m-1+t,\,m-1}x^t = \frac{1}{(1-x)^m} - \frac{1}{(1-x)^m} \sum_{t=0}^{t=m-1} c_{n+t,\,t}(1 - x)^t x^{n+1},$$

it represents in its second member, in a definite number of terms, the sum of the series in the first member, consisting of any number of terms. The first expression of the second member being invariant as to n, becomes the sum of the series when the second, dependent upon n, vanishes by special values of n or x. Thus, for $m = 2$, this second expression becomes zero when $x = (n + 2)/(n + 1)$; and this, substituted in (12), gives the identity:

$$(13) \qquad 1 + 2\left(\frac{n+1}{n}\right) + 3\left(\frac{n+1}{n}\right)^2 + \cdots + n\left(\frac{n+1}{n}\right)^{n-1} = n^2.$$

If the second expression in the previous formula becomes indefinitely small as n indefinitely increases, we have the summation of the infinite convergent series.

Written in the form:

$$\frac{1}{(1-x)^m} = \sum_{t=0}^{t=n} c_{m-1+t,\,m-1}x^t + \frac{1}{(1-x)^m} \sum_{t=0}^{t=m-1} c_{n+t,\,t}(1 - x)^t x^{n+1},$$

it represents the development of the binomial with negative exponents, *i.e.*, $(1-x)^{-m}$, into a finite series with a remainder term. The development is therefore good whether the series is finite, convergent or divergent.
If considered in the order:

$$\frac{1-\sum_{t=0}^{t=m-1} c_{n+t,\,t}(1-x)^t x^{n+1}}{(1-x)^m} = \sum_{t=0}^{t=n} c_{m-1+t,\,m-1} x^t,$$

it presents an extension of the theorem, that the difference of the same powers is divisible by the difference of the first powers.

Let, for illustration, $n = 3$, $m = 1, 2, 3, \cdots$, and we shall have successively:

$$\frac{1-x^4}{1-x} = 1 + x + x^2 + x^3,$$

$$\frac{1-x^4-4(1-x)x^4}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3,$$

$$\frac{1-x^4-4(1-x)x^4-10(1-x)^2x^4}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3,$$

etc.; where the ubiquitous binomial coefficients are recognized in their usual rôle.

IV. An Erroneous Rule for Finding a Hypotenuse, with a Corollary.

By M. W. Jacobs, Jr., Harrisburg, Pa.

To find the hypotenuse of a right-angled triangle, take half the length of the longer leg and add the length of the shorter leg. This was the rule nearly all the boys of an arithmetic class were found to be using. Much to the instructor's surprise, the correct answer was thus obtained to almost all of the examples. An inspection of the problems correctly solved in this manner showed that the ratio of the longer to the shorter leg was 4 : 3.

That this is the only ratio for which the boys' rule holds true may be demonstrated as follows: Let a = longer leg; b = shorter leg; h = hypotenuse. By the boys' rule

$$h = \frac{1}{2}a + b = \frac{a+2b}{2}.$$

Then

$$h^2 = \frac{a^2+4ab+4b^2}{4}.$$

But $h^2 = a^2 + b^2$. Hence

$$a^2 + b^2 = \frac{a^2+4ab+4b^2}{4}.$$

Then $4a^2 + 4b^2 = a^2 + 4ab + 4b^2$; whence $3a^2 = 4ab$; therefore $a/b = 4/3$. The

ratio a/b has the *single value* 4/3 (when in lowest terms); the rule holds for this but for no other value.

Corollary: Generalizing by substituting the rational fraction t/u in place of 1/2 in the boys' rule, we get:

$$h = \frac{t}{u}a + b = \frac{ta + ub}{u};$$

then

$$h^2 = a^2 + b^2 = \frac{t^2a^2 + 2tuab + u^2b^2}{u^2}.$$

Whence $u^2a^2 - t^2a^2 = 2tuab$, and $a(u^2 - t^2) = 2tub$.

Consequently

$$\frac{b}{a} = \frac{u^2 - t^2}{2tu}.$$

The two legs are therefore in the proportion: $(u^2 - t^2) : 2tu$. Let us take exactly the values $u^2 - t^2$ and $2tu$ for the sides.

Hence the hypotenuse $= \sqrt{(u^2 - t^2)^2 + (2tu)^2} = u^2 + t^2$. Therefore

$$a = u^2 - t^2; \qquad b = 2tu; \qquad h = u^2 + t^2, \tag{1}$$

where t and u are integers, to be chosen at pleasure, (except that $u > t$, since a is to be positive). More generally we may write

$$a = c(u^2 + t^2); \qquad b = c(2tu); \qquad h = c(u^2 + t^2), \tag{2}$$

which is the well-known formula for Pythagorean triangles in its most general form. The particular case, $c = 1$, gives the primitive types.

RECENT PUBLICATIONS.

REVIEWS.

History of the Theory of Numbers.

History of the Theory of Numbers. Volume I: *Divisibility and Primality.* By Leonard Eugene Dickson. Carnegie Institution of Washington, 1919, publication number 256, Vol. I. 14 + 486 pages. Price, in paper, $7.50; in cloth, $8.00.

There are at least three great ends to serve by the history of science: to enrich the general culture and intellectual life of cultivated people; to enable a scientific worker quickly to orient himself in a chapter of science so as to proceed most readily to a detailed mastery of its literature; to enable a scientific worker to ascertain with completeness what has already been attained in a given subject. To serve any one of these three purposes requires a very different

treatment from that needed in the case of the others. It is probably impossible to contribute satisfactorily at the same time and by the same work to any two of them. The present literature of each class is altogether inadequate in the case of science in general and in each particular discipline.

It is hard to say which of these three aspects of the need of scientific history presses hardest upon our attention. Each one would probably decide in favor of that which most vitally affects his own particular interest. Probably the last of the three purposes named is realized in the fewest instances. This work of Dickson, of which the first volume is now before us, embodies an attempt to serve this purpose adequately so far as the theory of numbers is concerned. The work is arranged so as to be suitable not only to the needs of the researcher but also to the interests of the amateur who studies number theory in the spirit of play and with restful delight.

From the preface we quote: "The theory of numbers is especially entitled to a separate history on account of the great interest which has been taken in it continuously through the centuries from the time of Pythagoras, an interest shared on the one extreme by nearly every noted mathematician and on the other extreme by numerous amateurs attracted by no other part of mathematics. This history aims to give an adequate account of the entire literature of the theory of numbers. The first volume presents in twenty chapters the material relating to divisibility and primality. The concepts, results, and authors cited are so numerous that it seems appropriate to present here an introduction which gives for certain chapters an account in untechnical language of the main results in their historical setting, and for the remaining chapters the few remarks sufficient to clearly characterize the nature of their contents."

To give an adequate account of the entire literature of so vast a subject and one of such long history as the theory of numbers is an undertaking of enormous magnitude; and it is carried through in this work with a marvelous success in the presence of which one must pause in admiration. Henceforth this history will be indispensable to all investigators in the theory of numbers. No one will be justified in publishing his researches on this subject till he has compared them with the relevant chapters of this work. Frequent duplications, all too common in the past development of the subject, will have far less excuse for their continuance. A great deal of scientific energy will be saved through this storehouse of exact information about the development of the theory of numbers and the actual present state of knowledge.

As indicated already, the subject is well suited to the interests of amateurs in mathematics. The reviewer's private correspondence shows that not a few such people, interested in number theory, are to be found throughout our country (and presumably in other countries as well). Many of these are far separated from adequate scientific libraries. To such this book will now be indispensable. With it in hand, even though they are separated from library sources, they will be able to get a fair estimate of the novelty of their work. The book will certainly add greatly to the delight and the value of the work of such people. Since

the theory of numbers is precisely that part of mathematics in which the amateur has the greatest opportunity of making a contribution of interest it may be hoped that one of the pleasing fruits of publishing such a work as the present will be the inspiration afforded to amateurs to pursue their work and to make known their discoveries in this field of fascinating interest which has inspired the enthusiasm of people of every century for some millenniums.

Where the detailed results are so numerous and are distributed widely through so varied a literature it is impossible that the first adequate research into the matter should uncover all the information and that no errors should be made in the presentation of what was found. Realizing this, the author has put in his preface the statement: "Readers are requested to . . . increase the usefulness of this work by sending corrections, notices of omissions, and abstracts of papers marked not available for report, for insertion in the concluding volume." It is to be hoped that this request may have wide notice and that every addition or correction found to be needed will be sent to the author at the University of Chicago. Besides several minor corrections which the reviewer is sending to the author in this way mention should now be made of the following changes needful to avoid confusion or annoyance on the reader's part: p. 131, l. 4, change "simple" to "prime power cyclic"; p. 381, l. 11, change k to m in two places; p. 403, l. 22, change $U_{2\ +1}$ to U_{2n+1}; p. 413, l. 13, read J. Perott[8] noted that, if $p_1, \cdots, p_n$ are all the primes and their product $\leqq N \cdots$.

Probably no one sufficiently interested in the theory of numbers to read this review will be satisfied without examining the volume itself. He will naturally begin with the ten-page preface or prefatory introduction, where we have a rapid review of some of the main features of the development of the theory covered by the present volume. Not a few people will wish that this had been more extended, especially since the immense amount of detail necessary in the body of the book makes it hard sometimes to select the salient matters. Necessarily the classification is not into rigidly separated portions so that matters more or less closely related will sometimes be found in different chapters. The subject index will be found to bring these together in a helpful way.

The book opens with a forty-eight page history of perfect, multiply perfect, and amicable numbers, a group of closely related topics which have engaged the attention of arithmeticians of every century of the Christian era. If the sum $\sigma(n)$ of the divisors of n is a multiple mn of n then n is said to be a multiply perfect number of multiplicity m. If $m = 2$, so that n is the sum of its aliquot divisors, we say that n is perfect. Twelve perfect numbers are known as such. The list of multiply perfect numbers is much larger. In comparing the two new multiply perfect numbers due to Cunningham and given on page 37 of Dickson's *History* with the table of Carmichael and Mason I discovered in the latter the error of taking 137561 [= 151.911] to be a prime. This requires the exclusion from their table of the 19 numbers which have the factor 19^4 (but not 19^5). But the two numbers of Cunningham show that if one of the numbers $C \cdot 19^2 \cdot 127$ and $C \cdot 19^4 \cdot 151 \cdot 911$ is multiply perfect and C is prime to $19 \cdot 127 \cdot 151 \cdot 911$ then the

other of these numbers is also multiply perfect of the same multiplicity. From this principle and the table of Carmichael and Mason it is easy to find (in a manner now obvious) 47 additional multiply perfect numbers. Moreover I have found also another number of multiplicity 7:

$$2^{32}\cdot 3^{11}\cdot 5^{4}\cdot 7^{8}\cdot 11^{2}\cdot 13^{2}\cdot 17^{2}\cdot 19^{3}\cdot 23\cdot 31\cdot 37^{2}\cdot 61\cdot 67\cdot 71\cdot 73\cdot 89\cdot 181\cdot 307\cdot 1063\cdot 2141 \cdot 599479.$$

In Carmichael and Mason's list the first number containing the factor 2^{46} should have the factor 67 inserted into it. With these additions and corrections there are now 282 known perfect and multiply perfect numbers.

Two numbers m and n are said to be amicable if their sum is equal to the sum of the divisors of each. Sixty-eight such numbers are known of which fifty-nine are due to Euler, three having been discovered before his time and six since. We propose to extend the notion of amicable numbers (in a way different from those indicated on page 50 of Dickson's *History*) and to say that two numbers m and n are multiply amicable of multiplicity k if k times their sum is equal to the sum of the divisors of each, so that we have $\sigma(m) = \sigma(n) = k(m + n)$. The usual amicable numbers are those for which $k = 1$. That multiply amicable numbers exist is shown by the following examples for $k = 2$:

$$2^{5}\cdot 3^{2}\cdot 7\cdot 13\cdot \begin{Bmatrix} 5\cdot 11 \\ 71 \end{Bmatrix} \qquad 2^{2}\cdot 3^{2}\cdot 5\cdot 7\cdot 13\cdot \begin{Bmatrix} 11\cdot 19 \\ 239 \end{Bmatrix}.$$

The last is one of eleven pairs of multiplicity 2 which may be obtained from this one by replacing the factor $2^2\cdot 7$ by any one of the eleven known perfect numbers other than 6. Other similar sets may also be made up by multiplying certain amicable odd numbers of Euler's list by perfect numbers. It would be rather interesting to have a systematic determination of multiply amicable numbers after the method employed by Euler for $k = 1$ and so well described by Dickson on pages 42–46. [A similar generalization of Dickson's amicable triples (p. 50) would also be of some interest.]

After a short chapter on the number and sum of divisors and certain related problems of Fermat and Wallis, we have the extensive third chapter (pp. 59–103) on Fermat's and Wilson's theorems, their generalizations and converses, and symmetric functions modulo p of the positive integers less than p and prime to p; and this is followed by a closely related short chapter on the residue of $(u^{p-1} - 1)/p$ modulo p. In the preface it is pointed out that it was while investigating perfect numbers that Fermat was led to the theorem which bears his name and which forms the basis of a large part of the theory of numbers. Numerous properties and generalizations of Euler's φ-function are given in the long chapter V (pp. 113–158). Owing to the frequent occurrence of the φ-function in many parts of number theory the reader should employ the index to find accounts of its use in other ways than in those mentioned in chapter V. In this connection special attention should be given to the chapters on Lucas's functions and on

recurring sequences of integers. The great amount of important detail included in such major chapters as III and V is such as to render an adequate summary impossible.

After an elementary chapter dealing with periodic fractions and factors of $10^n \pm 1$ we have in VII (pp. 181–222) and VIII (pp. 223–262) two other of the more fundamental chapters. The first of these deals with primitive roots and binomial congruences; the second with higher congruences. Together they form one of the most valuable sections of the book. Following them we have a chapter of sixteen pages on divisibility of factorials and multinomial coefficients. Then comes chapter X (pp. 279–325) on the sum and number of divisors of an integer and several closely related topics. Many of the results summarized here are closely connected with or are derived by means of various functional relations, sometimes associated with functions constructed for such use and at other times having to do with those which give rise incidentally to such applications. As in the case of other important chapters, its content is too vast for summarizing.

The next five chapters have to do with minor matters: miscellaneous theorems of divisibility, greatest common divisor, least common multiple; criteria for divisibility by a given number; factor tables, lists of primes; methods of factoring; Fermat numbers $F_n = 2^{2^n} + 1$. These chapters are necessary to completeness; more than that, some of the matter recorded still has its fascinating appeal.

In chapters XVI and XVII (pp. 381–411) we have an account of the factors of $a^n \pm b^n$, recurring series and Lucas's functions. The more or less fragmentary results here gathered together are of interest because of their intimate connection with the factorization of large numbers and the identification of large primes. Here the extent of our knowledge is so clearly marked by a definite line of cleavage between the known and the unknown, and the former is so clearly a mere fragment of the latter, that we contemplate our present state of knowledge with much the same feeling as that of the fabled Tantalus. The general theory of recurrent sequences of integers deserves a more penetrating analysis than it has yet received. It appears to be destined to bring out deep-lying connections among important parts of the theory of divisibility and primality. One who would go into this field for research could readily orient himself in it by means of Dickson's *History* and the two volumes (especially the second) of Bachmann's *Niedere Zahlentheorie*.

In Chapter XVIII (pp. 413–440), on the theory of prime numbers, we have in one respect, a departure from the plan followed in the others. That part of the subject covered by Landau's *Verteilung der Primzahlen* is passed over with a reference to its treatment in that work, particularly in the historical sections. Such other matter as pertains to the theory of primes is adequately treated in this chapter. To the various proofs that an infinitude of primes exists may be added the following: Since for a positive integer k we have $2^{2^n} \equiv -1$ or $(2^{2^n})^{2^k} \equiv (-1)^{2^k}$ or $2^{2^{n+k}} + 1 \equiv 2 \bmod 2^{2^n} + 1$ it follows (as is well known) that no two numbers of the form $2^{2^n} + 1$ have a common prime factor. Hence there are at least as many primes as numbers of this form and hence an infinite number.

One should consult the index for references to other tests for primes than those given in this chapter.

Chapter XIX (pp. 441–451) deals with a very interesting range of material under the heading: inversion of functions, Möbius's function $\mu(n)$, numerical integrals and derivatives. Important facts relating to inversion appear in other chapters and may be found through use of the index. Some of the inversion properties are very remarkable. A further systematic study of inversion would probably lead to novel results of interest. The relation $F(m) = \Sigma f(d)$, where d ranges over all the divisors of m, may be expressed by means of a Stieltjes integral in the form

$$F(m) = \int_0^m f(t) d_t \rho(m, t),$$

when $\rho(m, t)$ denotes the number of divisors of m not greater than t and $f(t)$ is a continuous function of t taking integral values when t is integral. The number-theoretic solution expressing f in terms of F leads to a solution in a special case of the foregoing Stieltjes integral equation. Other inversion problems may likewise be expressed by means of a numerical Stieltjes integral equation of the above or similar type. Is it likely that the inversion of numerical functions has already yielded all the results which would be brought to light by a systematic study of the inversion of numerical Stieltjes integrals formed in a manner now obvious, if these should be studied systematically under the guidance afforded by the existing extensive theory of ordinary integral equations?

The volume closes with thirteen pages on properties of the digits of numbers, at the same time one of the most elementary and one of the least interesting chapters of the book.

Not a few conjectured or empirical theorems are mentioned at various places throughout the work. These have interest as suggesting unanswered questions and as pointing out apparently simple matters which we are not at present able to treat satisfactorily. In this connection it is interesting to quote from Gauss: "The most beautiful theorems of the higher arithmetic have this characteristic that they are easily discovered by induction while their proof lies concealed and can be brought out only by a deeply penetrating investigation." And he adds that precisely this constitutes one of the greatest charms in number theory investigation.

Of the conjectured (unproved but not disproved) theorems in the present volume we call attention to the following:

1. If $2^n - 1$ is a prime p then $2^p - 1$ is a prime (pp. 22 and 24). This theorem, if true and proved, would yield an infinite sequence of identified primes; no such sequence is known at present.

2. If n is a prime of the form $24x + 11$ [$24x + 23$] and if $2^n - 1$ is composite, the least factor is of the form $24y + 23$ [$48y + 47$] (pp. 29, 30, 31).

3. If p is any number and a any divisor of $2^p - 1$, $a = 8m \pm 1$ not being of the form $2^n - 1$, then $2^a - 1$ is composite (p. 30).

4. Let $s(n) \equiv s^1(n)$ denote the sum of the divisors of n less than n and form $s^k(n) = s\{s^{k-1}(n)\}$. The set of numbers n, $s(n)$, $s^2(n)$, $\cdots$ may form a periodic cycle. When they do not it is conjectured that the set contains a prime and hence terminates (pp. 48–50).

5. Every prime $an^2 + p$ has a as a primitive root if a is a primitive root of p and p is a prime between $a/2$ and a (p. 186).

6. Let N be the greatest common divisor of all integers represented by a polynomial $f(x)$ with integral coefficients without a common factor. It is conjectured that $f(x)/N$ represents an infinitude of primes when $f(x)$ is irreducible (p. 333).

7. Every number of the following sequence is a prime (p. 376):

$$2 + 1, \quad 2^2 + 1, \quad 2^{2^2} + 1, \quad 2^{2^{2^2}} + 1, \quad \cdots.$$

8. Dickson asked if $a_i n + b_i$ $(i = 1, \cdots, m)$ represent an infinitude of sets of primes when certain named necessary conditions are satisfied (p. 417).

9. Every even integer is a sum of two primes (pp. 421–424). On pp. 424 and 425 are several related theorems: every odd number is the sum of an odd square and the double of a prime of the form $4n + 1$; every prime $4n - 1$ is the sum of a prime $4m + 1$ and the double of a prime $4h + 1$ [from which it would follow readily that every integer is a sum of four squares]; every even number is the difference of two (consecutive) primes in an infinitude of ways; every odd number greater than 3 is of each of the forms $p_1 + 2p_2$, $p_1 - 2p_2$, $2p_1 - p_2$, where p_1 and p_2 are primes; every multiple of 6 is the difference of two primes of the form $6n + 1$.

10. No three successive primes are in arithmetical progression unless one of them is 3 (p. 425).

11. Is there an unlimited number of sets of five consecutive odd integers of which four are prime (p. 426)?

12. If $n > 1$ there is at least one prime in each of the intervals $n(n - 1)$ to n^2, n^2 to $n(n + 1)$ (p. 435).

13. At least four primes lie between the squares of two consecutive primes each greater than 3 (p. 436).

14. The $(2m + 1)$th prime in order of magnitude (unity being counted as a prime) can be composed by addition and subtraction of all the smaller primes each taken once; the $(2m)$th prime can be composed similarly, except that the next earlier prime is doubled (p. 436).

It would be of interest to have some of these conjectured theorems verified for a considerable range of values; it is a service which may well be rendered by an amateur in number theory.

I have verified for primes less than 1300 the conjecture that a prime of the form $4n + 3$ is the sum of a prime $4m + 1$ and the double of a prime $4h + 1$; the number of representations is sometimes large and seems to have a tendency to increase with increasing primes $4n + 3$. I have also verified up to 1300 that primes of the form $16n + 5$ [$36n + 7$] may be represented in the form $4p + q$ [$6p + q$] where p and q are primes of the form $4k + 1$.

A work of such extent and importance as the volume under review cannot be adequately described in a short article. To give a summary is impossible, since its 500 pages are themselves a summary in a very compact but lucid style of the results of many hundred memoirs. It is a piece of work for which one cannot find a parallel in the whole of scientific history. In completeness it goes far beyond anything before attempted for a wide range of material; and the work is done with a skill and accuracy which excite admiration. All persons interested in the theory of numbers are laid under a deep debt of gratitude to Professor Dickson.

R. D. CARMICHAEL.

NOTES.

Teubner announces the publication (1918) of a *Lehrbuch der Differential- und Integralrechnung und ihre Anwendungen* in two volumes, by R. Fricke.

The *Journal of the Michigan Schoolmasters' Club* (Ann Arbor, 1919) contains the proceedings of the fifty-third meeting, May 28–29, 1918. These include the following papers: "Group recitations in mathematics" by L. D. Wines, 82–86; "The value of measurements to high school mathematics" by S. A. Courtis, 87–94.

The following new English books are announced: H. Lamb, *Infinitesimal Calculus*, third edition (Cambridge University Press)—L. Silberstein, *Elements of Vector Algebra* (Longmans)—F. Cajori, *History of the Theories of Limits and Fluxions from Newton to Woodhouse* (Open Court)—H. E. J. Curzon, *Elementary Mathematics*, I and II.

W. Ahrens has published (Berlin, 1918) a new book on mathematical recreations, entitled *Altes und Neues aus der Unterhaltungsmathematik*. It is said by a reviewer in the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht* to contain several new topics in this field, and to be written in a very interesting style.

Oldenburg of Munich has published (1918) the eighth edition of *Einführung in die Mathematische Behandlung der Naturwissenschaften*, by W. NERNST and A. SCHOENFLIES, which was intended primarily for chemists. It will be recalled that *The Elements of the Differential and Integral Calculus* by J. W. A. YOUNG and C. E. LINEBARGER (New York, Appleton, 1900) was based upon the second edition of this book (1898) which first appeared in 1895.

The forty-fourth volume (Jahrgang 1913) of *Jahrbuch über die Fortschritte der Mathematik* was completed in 1918 and contained 86 + 1210 pages,—slightly more than the previous volume—The first number of the eightieth volume of the *Mathematische Annalen* was published last August.

A third edition of *American Men of Science, A Biographical Directory*, edited by J. McK. CATTELL, is in course of preparation. The appeal for information

in *Science*, July 25, 1919, states that there should be included in the book the names of all those in North America who have made contributions to the natural and exact sciences.

Archivio de Storia della Scienza (Roma, Nardecchia) is the title of a new quarterly periodical (Aldo Mieli, editor), the first number of which appeared last March. One of its articles, (pages 39–47) "Per una storia delle mathematiche nel secolo XIX" by G. LORIA, seems to have been inspired by F. Cajori's address, on "Plans for a history of mathematics in the nineteenth century," in 1918, as retiring president of the Mathematical Association of America.

The Rice Institute Pamphlets, volume 1, no. 2, May, 1915, contains English translations of two lectures delivered at the opening of the Rice Institute in 1912: (1) "Henri Poincaré" by VITO VOLTERRA, 133–162 (portrait of Poincaré) (2) "Molecular theories and mathematics" by EMILE BOREL, 163–193. Reference has been already made in this MONTHLY (*1917*, 250) to other lectures by Volterra and Borel published in *The Pamphlets*. All of these lectures are reprinted from volumes 2 and 3 of *The Book of the Opening of The Rice Institute*, Houston, Texas.

As announced in our September issue, H. G. ZEUTHEN, professor emeritus of the University of Copenhagen and of the Ecole Polytechnique of Copenhagen, died on his eightieth birthday. Among the books of this Nestor of Danish mathematicians are: *Die Lehre von den Kegelschnitten in Altertum* (German edition 1886), *Geschichte der Mathematik im Altertum und Mittelalter* (German edition 1896; briefer treatment, 1912), *Geschichte der Mathematik im XVI. and XVII. Jahrhundert* (German edition, 1903), and *Lehrbuch der abzählenden Methoden der Geometrie* (1912) a development of his *Encyklopädie* article (1905, French edition, 1915). As co-editor with Heiberg there had been issued two of the projected twelve volumes of *Mémoires scientifiques de Paul Tannery*.

Portraits and sketches of Zeuthen may be found in *Matematisk Tidsskrift*, A, February, 1919, in *Acta Mathematica, 1882–1912, Table générale* (Upsala. 1913), and in E. F. S. Lund's *Danske malede portraeter*, vol. 7 (Copenhagen, 1900).

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF MATHEMATICS, volume 41, no. 3, July, 1919: "Invariants of differential geometry by the use of vector forms" by C. D. Rice, 165–182; "On certain saltus equations" by H. Blumberg, 183–190; "Investigations on the plane quartic" by Teresa Cohen, 191–211; "On surfaces containing two pencils of cubic curves" by C. H. Sisam, 212–224; "Modular invariants of a quadratic form of a prime power modulus" by J. E. McAtee, 225–242.

ANNALS OF MATHEMATICS, 2d series, volume 20, no. 4, June, 1919: "Relations between abstract group properties and substitution groups" by G. A. Miller, 229–231; "The complete quadrilateral" by J. W. Clawson, 232–261; "Triply conjugate systems with equal point invariants" by L. P. Eisenhart, 262–273; "On a system of linear partial differential equations of the hyperbolic type" by T. H. Gronwall, 274–278; "Some properties of circles and related conics" by J. H. Weaver, 279–280; "Integrals in an infinite number of dimensions" by A. J. Daniell,

281–288; "On the differentiability of the solution of a differential equation with respect to a parameter" by J. F. Ritt, 289–291; "Note on the derivatives with respect to a parameter of the solutions of a system of differential equations" by T. H. Gronwall, 292–296; "On Quaternions and their generalization and the history of the eight-square theorem. Addenda" by L. E. Dickson, 297.

ATLANTIC MONTHLY, volume 124, no. 3, September, 1919: "The Scandal of Euclid, a Freudian analysis" by S. Strunsky, 332–337.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 25, no. 10, July, 1919: "The April meeting of the American Mathematical Society," by F. N. Cole, 433–448; "Report on the theory of the geometry of numbers" by H. F. Blichfeldt, 449–453; "Applications of the geometry of numbers to algebraic numbers" by L. E. Dickson, 453–455; "Products of skew-symmetric matrices," by A. A. Bennett, 455–458; "On the first factor of the class number of a cyclotomic field" by H. S. Vandiver, 458–460; "Corrections and note to the Cambridge Colloquium of September, 1916" by G. C. Evans, 461–463; "Circle and sphere geometry" by H. S. White, 464–467 [Review of Coolidge's *Treatise on the Circle and Sphere* (Oxford, 1916)]; Review by C. N. Moore of Karpinski, Benedict and Calhoun's *Unified Mathematics* (New York, 1918), and Pringsheim's *Vorlesungen über Zahlen- und Funktionenlehre*, Band 1 (Leipzig, 1916), 467–470; Review by L. W. Dowling of Fazzari's *I numeri reali e l'equazione esponenziale* $a^x = b$ *per le scuole medie superiori* (Palermo, 1918), 470–471; Review by F. M. Morgan of Carey's *Infinitesimal Calculus* (London, 1917), 472–473; Review by E. W. Brown of *Annuaire du Bureau des longitudes pour l'an 1919* (Paris, 1918), 473; "Notes," 474–480; "New Publications," 481–483; "Twenty-eighth annual list of papers," 484–491; "Index," 492–499.

EDUCATIONAL REVIEW, volume 58, no. 3, October, 1919: "Reopening of the Ecole Normale Supérieure" by A. Cohn, 181–200 [The opening exercises occurred March 23, 1919].

HARVARD GRADUATES MAGAZINE, volume 27, June, 1919: "Edward Charles Pickering" by J. H. Metcalf, 516–520 and Portrait; "More Reminiscences of '66" by George Batchelor [Pages 531–532—Benjamin Peirce: "In a tablet in the Exposition in Paris at the foot of a list of twenty of the greatest mathematicians of the last two thousand years appeared the name of Benjamin Peirce, our senior professor of mathematics. In '66 he was giving instruction which nobody understood, to a select audience of students who were trying to become mathematicians. He would begin by asking questions and attending to the answers, and then as some new truth flashed upon his mind, he would forget his class and cover the blackboard with algebraic symbols.

"At this time, Thomas Hill was president of the college, a great mathematician with no practical ability as an administrator. Before coming to Cambridge, he was minister of the Unitarian Church in Waltham. It is recorded by Robert S. Rantoul that when Peirce caught sight of some new truth which he could not reduce to writing, he would hasten to Waltham and describe to Hill what was disturbing his mind. He was the only man in America who could understand Peirce, and having 'no such original inspirations to trouble him, could better express in words, the new proposition when at last he understood it,' 'and towards morning send him home to Cambridge with his problem stated on paper in his pocket, and his mind at rest.'

"In a parlor in Boston, I heard Peirce in his old age deliver his last address. With great positiveness, he affirmed his belief in immortalility. He said, 'All my life I have studied the stars at long range. I expect to study them at short range. I may ride on the tail of a comet.' He said also, 'I have conceived the idea of an algebra of universal thought, but I am no longer able to work it out, and must leave it to those who come after me.' Probably no living man knows what he meant by that proposition.

"Of students that he instructed for many years in college, he said, 'We have seen many young men of brilliant parts, of whom we expected great things as scientific investigators; but they never fulfilled the promise of their youth. After long deliberation, I have come to the conclusion that the reason they failed was because they had bad hearts. The student of science must be a truth-seeker, and a truth teller, and the man whose heart is bad cannot be either.' "

JAHRESBERICHT DER DEUTSCHEN MATHEMATIKER VEREINIGUNG, volume 27, nos. 9–12, September–December, 1918: Determinanten und symmetrische Funktionen" by C. Kosta, 161–165; "Ausgezeichnete Elemente projektiver Gebilde, die ineinander liegen, und Folgerungen für die Homologien" by R. Sturm, 166–175; "Herstellung von Polaren" by R. Sturm, 175–178; "Die Mehrdeutigkeit von Integralen" by M. Pasch, 178–179; "Ueber die Bedingung der Integrier. barkeit" by M. Pasch, 179–181; "Ueber die Anordnung von vier Punkten einer Geraden" by L. v. Schrutka, 182–184; "Ueber die Vertauschbarkeit der Differentiationsfolge" by H. Hahn,

184–188; "Inwieweit kann Vandermonde als Vorgänger von Gauss bezüglich der algebraischen Auflösung der Kreisteilungsgleichung $x^n = 1$ angesehen werden?" by A. Loewy, 189–195; "Gaston Darboux" by A. Voss, 196–217; "Festrede zum 20 Stiftungstage der Göttinger Vereinigung zur Förderung der angewandten Physik und Mathematik" by F. Klein, 217–228 (full page portrait of Klein); "Die Forderung der Entscheidbarkeit," 228–232; "Ueber die Erweiterung des Grenzbegriffs" by M. Pasch, 232–234; "Ueber Kurven gleichmässigster Krümmung" by W. Blaschke, 234–236; "Bemerkung zu: Ueber Kurven gleichmässigster Krümmung" by R. v. Mises, 236; "Zu Felix Kleins goldenem Doktorjubiläum am 12. Dezember, 1918", *63–64*; Review by Bieberbach of Gauss's *Werke*, Band 10_1 (Leipzig, 1917), *66–68*; Review by F. Bernstein of Ziehen's *Das Verhältnis der Logik zur Mengenlehre* (Berlin, 1917), *66–68*.

JOURNAL OF THE INDIAN MATHEMATICAL SOCIETY, volume 11, no. 2, April, 1919: The Indian Mathematical Society and its Founder, Mr. V. Ramaswami Aiyar (Portrait frontispiece), 41–45 ["Our Society has just completed the twelfth year of its existence"]; "Conics with vanishing Θ, Θ'" by R. Vythynathaswamy, 46–49; "Bernoulli's polynomials and Fourier's series" by G. R. Ranganathan, 50–56; "Legitimacy of ordinary complex numbers" by G. A. Miller, 57–59; "Astronomical notes," 59–61; Problems and solutions, 62–80.

MATHEMATICAL GAZETTE, volume 9, no. 140, May, 1919: "Bordered antilogarithms, Trigonometric logarithms to every two minutes. Natural functions on three pages. Tables of exact squares" by G. H. Bryan with the assistance of C. Williams, T. G. Greak, Violet James, and others, 333–352 [Quotation from "Explanation of the tables": "Although books of mathematical tables have appeared by the dozen in recent years, most of them show little or no evidence of originality in the matter of arrangement. One of the most common characteristics of such tables is that much space is taken up with unnecessary repetition, as when both logarithms and antilogarithms, squares and square roots, or sines and cosines of all circles *sic* from 0° to 90° are separately tabulated, while the information supplied in other directions is defective, as when logarithms of reciprocals are omitted and tables of squares fail to give the correct values of the squares of integers.

"The Bordered Antilogarithm Table was published in the *Mathematical Gazette* for December 1915, and several correspondents expressed a wish that other tables should also be drawn up embodying improvements in arrangement proposed by me. This has now been done, and I have to thank those associated with my name on the titles for the whole work of making the necessary tabulations. It is interesting to note that the method of tabulation introduced in the Bordered Antilogarithm Table has already been adopted in a set of tables published for use on board the training ship *Conway*."]; Review, by G. Greenhill, of A. Gray's *Gyrostatics and Rotational Motion* (London, 1918), 353–356; "Books received, contents of Journals, etc.," 357–360.

MATHEMATICS TEACHER, volume 11, no. 4, June, 1919: "Tests of mathematical ability—their scope and significance" by Agnes L. Rogers, 145–164 ["bibliography," pp. 163–164]; "Some suggestions for courses in mathematics for non-college preparatory students," 165–171 [report presented at the spring meeting of the Association of Teachers of Mathematics in New England, May 3, 1919—Chairman of Committee: C. E. Paddock]; "A statistical study in correlation of efficiency in secondary mathematics and efficiency in other high school branches" by Nelle L. Ingels, 172–176; "Some angles of the right triangle" by A. L. Booth, 177–181 ("note" by E. R. Smith); "A psychological basis for a system of education with applications to mathematics" by W. P. Webber, 182–195; "Philosophy and non-euclidean geometry" by F. A. Foraker, 196–198; "Mathematical tests:—their relation to the mathematics teacher" by J. H. Minnick, 199–205; "Book Reviews," "Notes and News," 206–209.

NATURE, volume 103, June 12, 1919: "The age of the stars" by H. Shapley, 284—June 19: "Question relating to prime numbers" by A. Mallock, 305 [Quotations: "It is well known that no algebraical formula can represent prime numbers only, and that primes can only be found by trials (which may be facilitated by algebraical processes). If the mth prime number, counting from unity be denoted by n, and if n is plotted in terms of m, it will be found that n is approximately represented by a formula of the type Am^p ($A = 3{\cdot}15$, $p = 1.133$, are close to the values of the constants). . . . Are there any investigations which give a reason for the tendency of n to approach a definite function of m, or as to the ultimate value of dn/dm when m increases without limit?"]—July 10: "Question relating to prime numbers" by G. N. Watson, 364–365 [Quotation: "An approximate expression for n is $m \log_e m$."]—July 17: "American astronomy", 394 [reprinted in *Science*, August 22.]

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volumes 5, no. 6, June, 1919:

"On the most general class L of Fréchet, in which the Heine-Borel-Lebesgue theorem holds true" by R. L. Moore, 206–210; "On a certain class of rational ruled surfaces" by A. Emch, 222–224—No. 7, July: "On the twist in comformal mapping" by T. H. Gronwall, 248–250 ["Note II on conformal mapping under aid of Grant no. 207 from the Bache Fund"]; "Groups involving only two operators which are squares" by G. A. Miller, 272–274; "Real hypersurfaces contained in Abelian varieties" by S. Lefschetz, 296–298.

PROCEEDINGS OF THE ROYAL SOCIETY, London, series A, volume 95, June, 1919: "Bertram Hopkinson (with portrait)" by J. A. E., xxvi–xxxvi. [Born 1874; killed in a flying accident August, 1918. When only 29 years of age he was appointed to the chair of mechanism and applied mechanics in his alma mater, the University of Cambridge. "Many will mourn him as a trusted friend, but only those who knew something of what he did in the war can have a right idea of the magnitude of the nation's loss."]

QUARTERLY JOURNAL OF PURE AND APPLIED MATHEMATICS, volume 48, no. 3, June, 1919: Calculation of eighteen more, fifty in all, Eulerian numbers from central differences of zero" by S. A. Joffe, 193–271; "A new solution of Waring's problem" by G. H. Hardy and J. E. Littlewood, 272–288.

SCHOOL AND SOCIETY, volume 10, July 19, 1919: "Educational research and statistics; practical uses of an algebra standard scale" by F. R. Cawl, 88–90. [Use of Hotz's *First Year Algebra Scales*, Teachers College, Columbia University, Contribution No. 90, 1918.]

SCHOOL REVIEW, volume 27, no. 6, June, 1919: "Concrete geometry in the Junior High School" by W. H. Fletcher, 441–457.

SCIENCE, new series, volume 49, June 20, 1919: "The airplane in surveying and mapping" by E. L. Jones, 572–582—June 27: "Revista matemática Hispano-Americana" by G. A. Miller, 608–609—Volume 50, July 4, "'Working up' in a swing" by A. T. Jones, 20–21 [First paragraph: "A child sitting or standing in a swing can 'work up' until he is swinging through a considerable distance. How is it possible for him, without touching his feet to the ground, to increase the extent of his swinging? As I do not recall ever seeing any discussion of this matter, the following note may not be out of place."]—July 11: "The discovery of calculus', by A. S. Hathaway ,41–43 [First sentence: "The writer desires to call attention to certain disclosures here pointed out for the first time, whose conclusions are decisive in the matter of the celebrated controversy between Newton and Leibniz, regarding the discovery of calculus."]—July 18: "The history of science and the American Association for the Advancement of Science" by F. E. Brasch, 66–68; "Working up in a swing" by V. Karapetoff and P. E. Klopsteg, 70–71 [References for the treatment of Mr. Jones's problem are given to: (1) E. J. Routh's *Dynamics of a System of Rigid Bodies*, vol. 1, art. 287 (examples of living beings, no. 6); (2) *Zeitschrift für physikalischen und chemischen Unterricht*, vol. 16, 1913, p. 16, and vol. 17, 1914, p. 27.]

SCIENTIA, volume 25, no. 5, May, 1919: "Le matematiche in Ispagna, ieri ed oggi. Parte prima: Dal secolo XVI alla metà del XIX" by G. Loria, 353–359 [French translation, supplément, 79–85]; Review by M. Davidson of Whitehead's *The Organisation of Thought* (London, 1917) and *Science and the Nation*, essays edited by Seward (Cambridge, 1917), 410–412.—No. 6, June: "Le matematiche in Ispagna, ieri ed oggi. Parte II^a: I matematici moderni" by G. Loria, 441–449 [French translation, supplément, 99–108]; Review by P. E. B. Jourdain of Byerly's *An Introduction to the Use of generalized Coordinates in Mechanics and Physics* (Boston, 1916), 499–500; Review by G. Scorza of Forsyth's *Lectures introductory to the Theory of Functions of two Complex Variables* (Cambridge, 1914) and Goursat's *A Course of Mathematical Analysis*, vol. 2, parts 1–2, translated by Hedrick and Dunkel (Boston, 1916–1917), 500–502; Review by A. Mieli of *L'ottica di Euclide* edited by Orio (Milano, 1918), 502–503. [In his second article on Spanish mathematics, in discussing advances of the science in the last third of the nineteenth century, Loria sketches the career of Josè Echegaray, known throughout the world, not as a mathematician, but as the popular author of dramatic productions for which a Nobel prize was awarded. He was born between 1832 and 1836 and died in 1916. He is the author of half a dozen mathematical works.]—Volume 26, no. 1, July: "Le matematiche in Portogallo; ciò che furono, ciò che sono" by G. Loria, 1–9 (French translation, supplément, 1–9); Reviews by G. Scorza of Blichfeldt's *Finite Collineation Groups* (Chicago, 1917), H. T. Hudson's *Ruler and Compasses* (London, 1916) and H. Hancock's *Elliptic Integrals* (New York, 1917), 47–49.—No. 2, August: "Origines et developpement de l'algèbre" by L. C. Karpinski, 84–101; "Il problema dei tre corpi" by R. Marcolongo, 102–112, supplément, 17–27; Review by G. Scorza of Bôcher's *An introduction to the study of integral equations* (2nd edition, Cambridge, 1914), Dickson's *Linear algebras* (Cambridge, 1914),

and, Hardy and Riesz's *The general theory of Dirichlet's series* (Cambridge, 1915), 147–150; Review by A. C. D. Crommelin of Plummer's *An introductory treatise on dynamical astronomy* (Cambridge, 1918), 150–152.

THE TÔHOKU MATHEMATICAL JOURNAL, volume 15, nos. 3–4, April, 1919: "Note on Laguerre transformations," 227–231; "On Steiner's problem of closure" by K. Yanagihara, 232–235; "On curves with monotonous curvature" by T. Hayashi, 236–239; "Note on determinants whose matrix is that of an orthogonant increased or diminished by matrix unity," by T. Muir, 240–245; "On the sign and magnitude of the coefficients in the Fourier series, the sine series and the cosine series" by K. Ogura, 246–260; "Tractories in the irreversible field of force on a surface" by K. Ogura, 261–277; "A theorem on power series" by Y. Okada, 278–279; "A theorem on limits" by Y. Okada, 280–283; "An extension of a theorem of Scheefer's" by Y. Uchida, 284; "Ueber eine Ungleichung für bestimmte Integrale" by M. Jujiwara, 285–288; "Casey's Theorem in Japanese Mathematics" by T. Hayashi, 289–296 (in Japanese); "On Mr. Ône's theorem" by K. Ôishi, 297–299; "Some theorems on limits" by S. Narumi, 300–313; "Einige Sätze den Grenz wert betreffend" by T. Kubota, 314–322; "Ueber summierbare Reihen und Integrale" by M. Fujiwara, 323–329; "On the indeterminate form ∞/∞" by T. Hayashi, 330–336 (in Japanese).

TRANSACTIONS OF THE ROYAL SOCIETY OF CANADA, volume 12, series III, Mathematical, physical and chemical sciences, December, 1918, and March, 1919: "Concerning the integrals of Leleuvre" by C. T. Sullivan, 171–184; "Rational plane unharmonic cubics" by A. M. Harding, 185–194.

AMERICAN DOCTORAL DISSERTATIONS.

R. W. BRINK, "Some integral tests for the convergence and divergence of infinite series." *Published under the title:* "A new integral test for the convergence and divergence of infinite series," *Transactions of the American Mathematical Society*, 1918, volume 19, pp. 186–204. (Harvard, 1916.)

L. R. FORD, "On rational approximations to an irrational complex number," *Transactions of the American Mathematical Society*, 1918, volume 19, pp. 1–42. (Harvard, 1917.)

M. T. HU, "Linear integro-differential equations with a boundary condition," *Transactions of the Ameriean Mathematical Society*, 1918, volume 19, pp. 363–407. (Harvard, 1917.)

W. E. MILNE, "On the degree of convergence of Birkhoff's series." *Portions published as:* "On certain asymptotic expressions in the theory of linear differential equations," *Proceedings of the National Academy*, 1916, volume 2, pp. 543–545; "Note on asymptotic expressions in the theory of linear differential equations," *Bulletin of the American Mathematical Society*, 1917, volume 23, pp. 166–169. (Harvard, 1915.)

C. E. WILDER, "Problems in the theory of ordinary linear differential equations with auxiliary conditions at more than two points." *Portions published as:* "Expansion problems of ordinary linear differential equations with auxiliary conditions at more than two points," *Transactions of the American Mathematical Society*, 1917, volume 18, pp. 415–442. (Harvard, 1915.)

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF ALBION COLLEGE, Albion, Mich. [*1918*, 354–357.]

No meetings of this club were held during the period of the S. A. T. C. The first meeting of the year 1918–19 was held on January 21, 1919, and meetings were held regularly every other week thereafter, or as nearly according to this program as possible. Ten men and twelve women constituted the membership of the club for the year. Lieutenants W. Clarke Dean '21, and Murrey Fox '19,

and Captain Jesse Campbell '19 returned to the club after having been in war service.

Officers for the year 1918–19 were: President, Carla Kennedy '20; vice-president, Esther Pearl '20; secretary-treasurer, Donald Alexander '20; program committee: the vice-president, the secretary-treasurer and Floyd Harper '20.

The usual program for each meeting is as follows: (1) roll-call, each member responding to his name by a very brief discussion of an assigned topic; (2) short talk of the evening, limited to ten minutes; (3) long talk, occupying 20 to 30 minutes; (4) general discussion of any of the topics of the evening; (5) critic's report (the critic is usually a student).

Programs for 1918–19 are indicated below [(1) roll-call, (2) short talk, (3) long talk]. The short talks for the year were biographical.

January 21, 1919: Organization and election of officers.

February 4: (1) "A mathematical current event;" (2) "Thales" by Thelma Sharp '21; (3) President's address, "Mathematics applied" by Carla Kennedy '20.

February 18: (1) "A mathematical symbol, its meaning and origin;" (2) "Pythagoras" by Gertrude Pratt '21; (3) "Development and use of determinants" by Floyd Harper '20.

March 11: (1) "Special curves, equation, history and use;" (2) "Life of Euclid" by Christine Niemann '21; (3) "My experiences as a teacher of high-school mathematics" by Margaret Courtright '19.

March 25: (1) "An original mathematical jingle." (2) "Archimedes" by Donald Herrick '21; (3) "Mathematics applied to war" by Lieutenant Clark Dean '21.

April 22: (1) "Some interesting application of mathematics:" (2) "Napier" by Edgar Smith '21; (3) "Mathematics and anti-mathematics" by Vera Junkin '19.

May 6: (1) "Some desirable characteristics that I observed in my high-school teacher of mathematics;" (2) "Descartes" by Esther Pearl '20; (3) "Indeterminate forms in trigonometry" by Esther Pearl '20.

May 20: (1) None; (2) None; (3) "The relation of the teacher to the community" by Elizabeth Gordon '21.

General discussion: "How can a teacher best enter into the community life" led by Miss Gordon.

June 3: Social evening and election of officers. The officers elected for 1919–20 are: President, Esther Pearl '20; vice-president, Joyce Hadaway '20; secretary-treasurer, Almyra K. Priest '20; member of program committee, Elizabeth Gordon '21.

TOPICS FOR CLUB PROGRAMS.

16. Codes and Ciphers.

The devising of codes and ciphers and of methods for solving them is closely allied to mathematics. In fact, since every system of secret writing must be

based upon a certain fundamental set of symbols or sounds, all problems of deciphering such messages are essentially problems in permutations and combinations. The interest in the subject for students of mathematics is doubtless as much in the challenge to mathematical mastery as to the fascination of mystery.

The art of occult writing dates back to the ancient Greeks. The Spartans are credited with having sent messages by tattooing them upon the bodies of slaves. Sometimes, for greater secrecy, a slave whose scalp had been shaved and tattooed was not allowed to depart until his hair had grown long enough to conceal completely the message. When the slave had reached his destination the shaving of his scalp again revealed the message. The first record of secret writing on parchment, however, is probably that given by Plutarch in his life of Lysander where he gives[1] the following description of the manner of transmitting to that commander the order for his return:

> This scroll is made up thus: when the Ephors send an admiral or general on his way, they take two round pieces of wood, both exactly of a length and thickness, and cut even to one another; they keep one themselves and the other they give to the person they send forth; and these pieces of wood they call *scytales*. When, therefore, they have occasion to communicate any secret or important matter, making a scroll of parchment long and narrow like a leathern thong, they roll it about their own staff of wood, leaving no space void between, but covering the surface of the staff with the scroll all over. When they have done this, they take off the scroll, and send it to the general without the wood. He, when he has received it, can read nothing of the writing, because the words and letters are all broken up; but taking his own staff, he winds the slip of the scroll about it, so that this folding, restoring all the parts into the same order that they were in before, and putting what comes first into connection with what follows, brings the whole consecutive contents to view round the outside. And this scroll is called a *staff*, after the name of the wood, as a thing measured is by the name of the measure.

From Lysander's time (about 400 B.C.) to the present, great use has been made of secret writing by government officials in various connections of which the military has been the most important. A considerable number of special works on the subject have been written[2] but with the possible exception of Klüber's work, which seems to be the most widely known, these are not very likely to be available to most American students. The most available source for a brief discussion of the development of ciphers is likely to be the articles "Cryptography" by John Eglinton Bailey in recent editions of the *Encyclopædia*

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[1] *Plutarch's Lives*, the translation called Dryden's, corrected from the Greek and revised by A. H. Clough, in 5 vols. (Boston, Little, Brown & Co., 1905), Vol. III, pp. 125–126. A parallel rendering of the same passage is found in John and William Langhorne's translation (New York, 1858), p. 314.

[2] Some of those most frequently cited are:

Trithemius, John, *Polygraphia*, cited as published in 1500 and in various later editions.

Portae, J. B., *De furtivis literarum notis*, Neapoli, 1563, and 1602, Londoni, 1591; another edition: *De occultis literarum notis*, Montisbeligardi, 1593, Argentorati (Strassburg), 1603 and 1606. Discusses different modes of secret writing, of which one hundred and eighty are explained, and a method proposed by which they can be multiplied ad infinitum.

Schott, Kasper, *De Magia Universali*, Würzburg, 1676.

Klüber, J. L., *Kryptographik*, Tübingen, 1809.

Fleissner, *Handbuch der Kryptographik*, Vienna, 1881.

Delastelle, F., *Cryptographie*, Paris, 1902.

Brittanica and similar articles in several other encyclopædias. A classification of ciphers and a considerable number of interesting examples are given by Ball in his *Mathematical Recreations and Essays*.[1] But the most interesting illustrations and historically important examples are contained in various articles[2] printed in periodical magazines to be found in nearly every university or large public library.

Ciphers and codes are both based upon substitutions, the distinction being in the unit chosen as the basis of the substitution. A cipher is based upon the substitution of symbols for letters and a code is based upon the substitution of words or symbols for words. For example the two lines

a b c d e f g h i j k l m n o p q r s t u v w x y z
+ ∧ % * = / $ ÷ π ? d h 4 k b v 6 8 0 ( ) √ l a 1 t

could be made the basis of a cipher by agreeing that each letter in the upper line should always be represented by the character immediately beneath it in the second line. In this simple cipher the message "The Americans are coming" would read "(÷= +4=8π%+k0 +8= %b4πk$". Such a cipher could be improved by the further agreement that any character not appearing in the second of the two lines forming its basis should be considered as a "dead" or "null" character. By the use of these "null" characters as space fillers the division into words, which is of great aid in deciphering, could be avoided. However, any considerable message written in so simple a cipher as the above would be read in a very few minutes by an expert, since the normal relative frequency of occurrence[3] of letters in English is:

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[1] Fifth ed. (London, 1911), pp. 395–423; fourth ed. (1905), pp. 293–322.

[2] Of American articles appearing before the recent great war the following may be cited:

Century Magazine, Vol. 63, pp. 83–92, "Secret Writing, the Ciphers of the Ancients and some of those in Modern Use," by John H. Haswell.

Harper's Monthly, Vol. 97, pp. 105–109, "A Rebel Cipher Dispatch" by David Homer Bates. The author was manager and cipher-operator for the U. S. Military Telegraph Corps at Washington from 1861 to 1865. He describes ciphers used on both sides during the Civil War. Much other interesting information is given by the same author in an article entitled "Lincoln in the Telegraph Office," *Century Magazine*, Vol. 74, pp. 290–306.

North American Review, Vol. 128, 315–325, "Cryptography in Politics," by John R. G. Hassard. Gives examples of various ciphers and discusses some of the cipher telegrams connected with the presidential election of 1876. Other information concerning these cipher telegrams is given in the *Nation*, Vol. 27, p. 234, p. 250 and Vol. 28, p. 112.

Cosmopolitan, Vol. 36, pp. 475–478, 584–590, 715–718, three excellent articles on "Cryptography" by George Wilkes. Gives many examples of interesting and historical ciphers.

Bookman, Vol. 28, pp. 450–451, "Poe and Secret Writing," by Firman Dredd.

[3] The table given is from an article "Enciphering and Deciphering Codes" by Frank Morgan, of Fort Leavenworth, *Scientific American*, Vol. cxiii, No. 8 (August 21, 1915), p. 159. After giving an example of solving a much more difficult cipher message Mr. Morgan says: "We may say here that all practicable military ciphers are so marked by characteristic arrangement of the letters that no reasonably expert cipher man will be long in doubt as to the general method of enciphering. . . . A single short message would give trouble but military messages generally have fifty or more words."

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
20	4	8	11	33	6	5	16	17	1	2	10	8	19	21	6	¼	18	17	23	8	3	5	1	5	1.

As an illustration of code,[1] suppose that a German spy and his correspondents agree upon the following substitutions:

a = the
French ship = market
sailed from New York = price
sailed from Boston = quotation
to-day = is
for Marseilles = any even number
for Bordeaux = any number with a fraction.

The intelligence that "A French ship sailed from New York to-day for Marseilles" in this code becomes the innocent business message "The market price is 110." By a slight change of wording, "The market quotation is 110½" carries the information that "A French ship sailed from Boston to-day for Bordeaux."

Curiously enough, ciphers, which have been so widely used for secret writing and which were invented for that purpose, are not as securely secret as codes, which owe their development to their usefulness and economy in cabling commercial messages. Ciphers will, however, continue to be used where only *temporary* secrecy is desired, because, as Mr. Strother says, "cipher messages can be written and translated (by one's correspondent) without any equipment, like a code book, and much more rapidly than code. Thus, if a general in the field wishes to send a message ordering a colonel to advance in two hours, he sends it in cipher, because it would take the enemy more than two hours to decipher the message even if he intercepted it immediately, and because after the two hours have elapsed the information in the message would be of no value to him."

The recent great war not only led to renewed interest[2] in secret writing but also to the development of new forms. German spies made large use of geometric figures, drawings, sketches and pictures[3] as well as ciphers, codes and enciphered

[1] Cf. Strother, French, "German Codes and Ciphers," *World's Work*, June, 1918, p. 144.

[2] The following articles have appeared in the *Scientific American* during the war:

Brewton, William W., "The Science of the Cipher and an Explanation of Bacon's Undecipherable System," *Scientific American Supplement*, Vol. LXXIX, No. 2059 (June 19, 1915), pp. 394–395.

Edwards, E. C., "Cipher Codes and their Uses," *Sci. Am.*, Vol. CXIII, No. 1 (July 3 1915), p. 9.

Morgan, Frank, "Enciphering and Deciphering Codes," *Sci. Am.*, Vol. CXIII, No. 8 (August 21, 1915), p. 159.

Paddock, Ira J., "Cipher Codes Simplified," *Sci. Am.*, Vol. CXIII, No. 13 (Sept. 25, 1915), p. 271.

Woodworth, H. S., "A Simple Cipher Code," *Sci. Am.*, Vol. CXIII, No. 14 (Oct. 2, 1915), p. 291. Described a code adaptable to use with a slide rule.

Berkel, Ernest, "A Telegraphic Cryptogram," *Sci. Am.*, Vol. CXIII, No. 24 (Dec. 11, 1915), p. 519. Gives a convenient method for using figures in telegraphic messages.

[3] See in this connection the following:

Post, Melville Davisson, "German War Ciphers," *Everybody's Magazine*, June, 1918, pp. 28–34. Gives examples of diagrams and sketches. According to Mr. Post, the victory at the Marne was probably due, in part, to the amazing good fortune

codes. In spite of the fact that the Allies controlled all overseas mails and cables, the German Government was apparently able to communicate at will, for a time at least, with its embassies and agents everywhere.

It would seem as if no article on this subject should omit a reference to the work of John Wallis, who has been called by D. E. Smith[1] "one of the world's greatest decipherers of cryptic writing," or to Francis Bacon's famous biliteral cipher[2] and Ignatius Donnelly's remarkable attempt[3] to prove, by means of a cipher, that Bacon was the author of Shakespeare's works.

Readers of Edgar Allan Poe's "Gold Bug" are probably aware of his belief that every cipher could be resolved and his challenge[4] to his readers to submit any which he could not decipher.

If any of our readers have never read Conan Doyle's "Adventures of the Dancing Man" a mention of it in this connection may lead to the enjoyment of its delightful mystery.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about problems to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

2788. Proposed by WARREN WEAVER, University of Wisconsin.

Out of a freshman class of one thousand, various quiz sections are chosen by chance. After a semester's work A is found to stand 12th in a section of 18 students, and B is found to stand 18th

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of the French war office in obtaining, by an extraordinary accident, which can not now be made public, the key to the German radiograph cipher code. "General Joffre knew the German orders wherever the radiograph signal was used in the advance on Paris. And it is possible that the knowledge of the radiograph code enabled the American Government to decipher the messages sent from the great wireless station at Nauen to Count von Luxburg, and so to make public the vast system of German intrigue that has amazed the world" (p. 29).

Strother, French, "Fighting Germany's Spies," *World's Work*, June, 1918, pp. 134–153. Gives descriptions and examples of the "Playfair" cipher (used by the British army in the field and pronounced by Mr. Strother "the cleverest transposition cipher ever devised"), the "Chess Board" cipher and the Bolo Pasha code.

Literary Digest, "A Graphic Spy Code," Vol. 56, p. 38, and "Keeping Government Cipher-Codes Secret," Vol. 51, pp. 546–548.

[1] Cf., Smith, D. E., "John Wallis as a Cryptographer" *Bulletin of the American Mathematical Society*, Vol. 24, No. 2 (Nov. 1917) pp. 82–96.

[2] *The Works of Francis Bacon, Baron of Verulam, Viscount of St. Alban, and Lord High Chancellor of England*, collected and edited by James Spedding, Robt. Leslie Ellis and Douglas Denon Heath (London, 1879), Vol. I, pp. 659–661. In a note on p. 843 Spedding credits the idea of a biliteral cipher, which Bacon seems to claim as his own, to John Baptist Porta (cf. *l.c.* above) although employed by him in a different manner.

[3] Donnelly, Ignatius, "The Great Cryptogram," R. S. Peale & Co., Chicago, N. Y., London, 1888.

[4] *The Works of Edgar Allan Poe*, with an introduction and a memoir by Richard Henry Stoddard (A. C. Armstrong & Son, N. Y., 1884), Vol. I, pp. 431–451. This essay on "Cryptography" contains, according to the editor, all that Poe had to say of importance on the subject. The challenge, which was first printed in *Graham's Magazine* for July, 1841, is reprinted on p. 442.

In Stedman and Woodberry's edition of Poe's works (Chicago, 1896) the essay on "Cryptography" appears in volume 9, pp. 260–278.

in a section of 27 students. Show that, had the whole class of one thousand taken the work in one large section, the most probable result is that B would have finished the semester's work six places ahead of A, and that the expectation of B's lead on A is 11.

2789. Proposed by KURT LAVES, University of Chicago.

Given a quadrilateral $ABCD$ for which $AC + BC < AD + BD$ to construct, by means of the ruler and compass only, the pair of tangents from D to the hyperbola (ellipse) for which A and B are the foci and C a point on the hyperbola (ellipse).

2790. Proposed by J. W. LASLEY, JR., University of North Carolina.

How shall we buy 12 eggs for 80 cents, if eggs sell as follows: hen eggs at 5 cents each, duck eggs at 7 cents each, and turkey eggs at 8 cents each, provided we buy some of each?

2791. A cup of wine is suspended over a cup of equal capacity full of water; through a small hole in the bottom, the wine drips into the water, and the mixture drips out at the same rate. When the wine cup is empty, what part of the contents of the lower cup is water? [Proposed by Charles Gilpin, Jr., Philadelphia, as Problem 287 in *The Mathematical Visitor*, January, 1881, volume 1, page 193. No solution was published in the *Visitor*.]

2792. Proposed by B. J. BROWN, Kansas City.

Solve the differential equation,

$$x^2(1-x)\frac{d^2y}{dx^2} + 2x(2-x)\frac{dy}{dx} + 2(1+x)y = x^2.$$

461 (Algebra) [June, 1916]. Proposed by E. T. BELL, University of Washington.

(1) Two events have probabilities p, q respectively. The events may be either (i) mutually independent; or (ii) mutually exclusive. Assign meanings to the symbol p^q, in terms of the two events where p^q is written for $p \times p \times \cdots \times p$, ($q$ factors p), in cases (i), (ii), and $p \times p$ has the customary meaning (as a probability).

(2) What relations, if any, other than (i) and (ii) can exist between two events? Upon what postulates is the answer to this based?

463A (Geometry) [May. 1915]. Proposed by B. J. BROWN, Kansas City.

If μ and ν are the parameters of the two confocal conics through any point on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

show that $\mu + \nu + a^2 + c^2 = 0$, along a central circular section.

470 (Geometry) [September, 1915]. Proposed by R. E. MORITZ, University of Washington.

Prove that

$$\theta = (\lambda + (q/p)\mu)\pi, \qquad (\lambda = 1, 2, 3, \cdots, q-1;\ \mu = 0, 1, 2, \cdots, p-1),$$

and

$$\theta = (2\lambda - 1)\pi/2 + (q/p)(4\mu \pm 1)\pi/2, \qquad (\lambda = 1, 2, 3, \cdots, (q-1)/2;\ \mu = 0, 1, 2, \cdots, p-1),$$

determine the same set of points on the curve $\rho = a \cos (q/p)\theta$, where p and q are two odd integers without a common factor, and a is any constant.

499 (Geometry) [November, 1916]. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

Find the surfaces all the plane sections of which are circles.

501 (Geometry) [November, 1916]. Proposed by R. P. BAKER, University of Iowa.

Find the minimum amount of lumber one inch thick required to pack a gross of spheres three inches in diameter in a rectangular box.

SOLUTIONS OF PROBLEMS.

430 (Algebra) [March, 1915]. Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve the following equations algebraically and graphically:

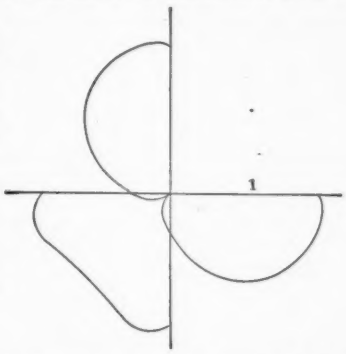
$$x^y + y^x = xy, \qquad x^x + y^y = x + y.$$

SOLUTION BY A. A. BENNETT, University of Texas.

There is no pair of real values, with the doubtful exception of (0, 0), which furnishes a simultaneous solution to the two equations given.

The equations being transcendental no "algebraic" solution is attempted. The discussion here is analytical, supplemented by a graph.

In examining the second of the above equations, it is convenient to distinguish three types of real values, viz., (*A*), the type in which the variable has any positive real value; in both (*B*) and (*C*) are considered only negative and rational numbers when written as irreducible fractions; in (*B*) the numerator of such fractions is even, while in (*C*) it is odd. Irrational negative values or rational negative values which when reduced to lowest terms have even denominators cannot appear among real solutions.



The following graph is liable to misinterpretation and will be qualified.

In the first quadrant there is a single point, (1, 1).

In the second quadrant, the graph is discontinuous, x being of type (*C*) and y of type (*A*). An infinite number of points certainly exist satisfying the equation.

In the fourth quadrant, x is of type (*A*) and y of type (*C*). Above remarks apply by symmetry.

In the third quadrant there are three branches to consider: (i) an apparent continuation of the branch of the second quadrant, (ii) an apparent continuation of the branch of the fourth quadrant, (iii) a separated branch containing $(-1, -1)$. Untractable questions of irrationality make doubtful the existence of any actual solutions in the third quadrant other than $(-1, -1)$. If any exist, they are on the branches above noted and are of the following types. On (i) x is of type (*C*) and y of type (*B*), on (ii) x is of type (*B*), and y of type (*C*); on (iii) x and y are each of type (*C*).

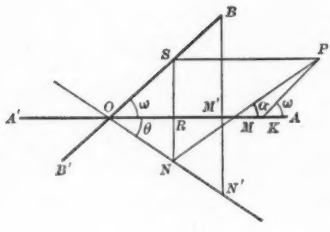
A real evaluation of $f(x, y) \equiv x^y + y^x - xy$ is only possible for both x and y rational, so far as points of the above graph are concerned. A study of $f(x, y)$ on the points of the graph shows it, even if existent, very far from zero, except as the origin is approached on branches (i) and (ii) in the third quadrant.

519 (Geometry) [September, 1917]. Proposed by OTTO DUNKEL, Washington University.

Given the conjugate axes $A'OA$ and $B'OB$ of an ellipse, points of the curve may be constructed as follows: Drop the perpendicular BM' to OA and produce it to N' so that $BN' = AO$. Draw a straight line through O and N'. Upon a straight edge, say that of a slip of paper, the points N, M, and P are marked so that $NM = N'M'$ and $MP = M'B$. Place the straight edge so that N falls on ON' and M on OM' and mark the position of P. This gives a point of the ellipse and by sliding the straight edge into new positions other points may be rapidly obtained. If the axes are perpendicular this gives the familiar trammel construction. Prove the correctness of this construction.

I. SOLUTION BY F. H. SAFFORD, University of Pennsylvania.

Let the axes of coördinates be taken along OA and OB, calling OA and OB, a and b, respectively. Then for P, x is OK and y is KP. From the given conditions, $MP = M'B = b \sin \omega$; hence, from the triangle MKP, (if $\alpha = \angle KMP$),



(1) $y = KP = MP \sin\alpha/\sin\omega = b \sin\alpha.$

Also $NM = N'M' = a - b \sin\omega$, and from the triangle ONM, $OM = NM\ (\sin\alpha \cot\theta + \cos\alpha)$. From triangle $ON'M'$, $\cot\theta = OM'/N'M' = b \cos\omega/NM \therefore OM = b \sin\alpha \cos\omega + NM \cos\alpha = a \cos\alpha - b \sin(\omega - \alpha)$. From triangle MKP, $MK = MP \sin(\omega - \alpha)/\sin\omega = b \sin(\omega - \alpha)$. Hence,

(2) $x = OM + MK = a \cos\alpha.$

Thus (1) and (2) are the desired (parametric) equations of the locus of P.

II. Remarks by the Proposer.

The part of the above proof following (1) can be shortened somewhat thus: Draw the straight line NRS perpendicular to OA, and join P to S. Then $NM/NP = N'M'/N'B = NR/NS$ and hence RM is parallel to SP and also $x = OK = SP$. It now follows that $x = PN \cos\alpha = a \cos\alpha$.

The problem may also be treated geometrically as is done in Rouché and Comberousse, *Traité de Géométrie*, deuxième partie, 8e éd., 1912, pp. 341–345.

332 (Mechanics) [October, 1916]. Proposed by E. E. MOOTS, University of Arizona.

A correct wording of this problem is given in 490 (Geometry) [May, 1916], a solution of which, by A. M. Harding, was published in February, 1917.

198 (Number Theory) [November, 1913; June, 1919]. Proposed by the late ARTEMAS MARTIN.

Prove that every even number is the sum of two prime numbers.

Note by R. C. Archibald, Brown University.

This is Goldbach's empirical theorem and the conjecture appears in a letter to Euler dated June 7, 1742 (*Corresp. Math. Phys.*, ed. Fuss, Vol. 1, 1843, p. 127). The first published statement of the theorem was by E. Waring in his *Meditationes Algebraicæ;* 1770, p. 217. E. Haussner verified the law for numbers up to 10000 (*Jahresbericht der Deutschen Math. Verein.*, Vol. 5, 1896, 62–66), and E. Maillet proved that every even number $\leqq 350000$ (or 10^6 or $9 \cdot 10^6$) is, in default by at most 6 (or 8 or 14), the sum of two primes (*L'Intermédiaire des mathématiciens*, volume 12, 1905, p. 108). These notes are taken from L. E. Dickson's *History of the Theory of Numbers*, volume 1, 1918, where the complete history of the theorem may be found on pages 421–425. No proof of this theorem has yet been discovered.

201 (Number Theory) [December, 1913; June, 1919]. Proposed by E. T. BELL, University of Washington.

Eisenstein proposed (*Crelle*, t. 27, p. 282) as the simplest of several problems: "In the expansion of

$$\frac{1 + z + z^2 + \cdots + z^{p-1}}{(1 - z)^{p-1}} - 1,$$

where p is prime, to show that the coefficients of the various powers of z are all divisible by p."

Solution by R. C. Archibald, Brown University.

$$\frac{1 + z + z^2 + \cdots + z^{p-1}}{(1 - z)^{p-1}} - 1 = \frac{1 - z^p}{(1 - z)^p} - 1,$$

$$= \left(pz - \frac{p(p-1)}{1 \cdot 2} z^2 + \cdots + \frac{p(p-1)}{1 \cdot 2} z^{p-2} - pz^{p-1} \right) (1 - z)^{-p}.$$

Multiplying the first factor of this product by the second factor, expanded, we have the desired result. For, in each factor the coefficients are integers, and p is contained in every coefficient of the first factor.

Also solved by P. J. da Cunha, A. Pelletier, and Elijah Swift.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

Dr. W. V. N. GARRETSON has been appointed assistant professor of mathematics in Rutgers College.

Dr. R. L. CHARLES has been promoted to an associate professorship of mathematics in Lehigh University.

Assistant professor J. D. BOND, of the mathematics department of the Agricultural and Mechanical College of Texas, has been granted a year's leave of absence and will study at the University of Michigan where he will hold a university fellowship.

Mr. E. T. BROWNE has accepted a fellowship in mathematics at the University of Chicago for the current academic year.

Mr. WARREN WEAVER has returned to Throop College, Pasadena, Cal., after spending the past year as instructor in mathematics at the University of Wisconsin.

Lieutenant FREDRICK WOOD has returned from service overseas to the University of Wisconsin as instructor in mathematics.

Lieutenant F. L. BROWN, of the meteorological service, has been appointed instructor in mathematics at Northwestern University.

Dr. ARNOLD DRESDEN has resumed duties as assistant professor of mathematics at the University of Wisconsin, after a year of service with the Red Cross in France.

During the war Dr. E. J. MOULTON, associate professor of mathematics at Northwestern University, had charge of the Dearborn Observatory and the astronomy department of the University, in addition to his regular duties in the mathematics department.

Dr. C. R. DINES, formerly assistant professor of mathematics at Dartmouth College, has been in the Federal Reserve Bank of Chicago for the past year.

At Dartmouth College, Professor J. W. YOUNG is on leave of absence, for the current academic year. He is residing at Hanover, N. H., working on the National Committee on Mathematical Requirements, of which he is chairman and treasurer. Professor R. D. BEETLE has been appointed chairman of the mathematics department, in the absence of Professor YOUNG. Assistant Pro-

fessor F. M. MORGAN has been appointed Freshman Class Officer of the College. Professor L. L. SILVERMAN has resigned to enter a banking house in Boston.

At Northwestern University, Dr. H. B. CURTIS, formerly of Barnard College, has accepted a position as instructor in mathematics.

Miss M. E. DANIELLS has been promoted from an instructorship to an assistant professorship of mathematics at Iowa State College.

Professor F. F. DECKER, of the department of mathematics of Syracuse University, has been made director of the evening session of that institution.

M. E. GRABER, professor of mathematics at Heidelberg University, has been appointed professor of physics at Morningside College, Sioux City, Iowa.

Dr. T. H. GRONWALL has been appointed mathematics and dynamics expert on the technical staff of the ordnance office, of the War Department, Washington. The University of Texas has granted leave of absence to Professor A. A. BENNETT to carry on work in this same office during 1919–20.

E. V. HUNTINGTON, associate professor of mathematics in Harvard University, has been promoted to a full professorship of mechanics. His teaching activities will be divided as heretofore between the Division of Mathematics and the Division of Engineering.

Professor LAURENCE HADLEY of Earlham College has resigned to accept a position as associate professor of mathematics at Purdue University.

Mr. M. F. JOHNSON, late of Michigan Agricultural College, has been appointed instructor of mathematics at the University of Michigan.

Dr. E. A. KIRCHER has resigned his appointment as instructor of mathematics at Yale University to enter business.

Professor G. H. LING, dean of the Faculty of Arts of the University of Saskatchewan, has been appointed acting-president of the University during a three months vacation of the president.

Assistant Professor C. N. MILLS, of South Dakota State College, has been elected professor of mathematics at Heidelberg University, Tiffin, Ohio.

F. W. PARSONS of Ohio Northern University has been made assistant professor of mathematics and engineering in Whittier College.

Dr. T. M. PUTNAM, professor of mathematics and dean of the undergraduate division in the University of California, has been appointed acting dean of the college of letters and science in the place of the late Professor H. M. STEPHENS.

Assistant Professor W. W. RANKIN, of the University of North Carolina, has been granted leave of absence to serve for the year 1919–20 as instructor in mathematics at Columbia University.

At the University of Wisconsin Dr. EUGENE TAYLOR has been appointed assistant professor of mathematics and Miss MARY A. COLPITTS instructor in mathematics.

F. C. TOUTON, formerly of the St. Joseph (Mo.) Junior College, received the degree of doctor of philosophy last June from Columbia University and has been appointed supervisor of high schools for the state of Wisconsin.

Assistant Professor M. O. TRIPP, of the University of Maine, has been promoted to an associate professorship of mathematics.

Dr. E. S. ALLEN, of the University of Michigan, has been appointed assistant professor of mathematics at West Virginia University.

Dr HENRY BLUMBERG, recently associate in mathematics at the University of Illinois, has been promoted to an assistant professorship of mathematics.

Mr. D. C. KAZARINOFF, recently of Carleton College, has been appointed instructor in Mathematics at the University of Michigan.

Mr. A. D. CAMPBELL remained as instructor in mathematics at Cornell University instead of accepting the appointment to Yale University announced in the MONTHLY for June.

V. J. BOUSSINESQ, professor of calculus of probabilities and mathematical physics at the University of Paris, has retired as honorary professor after twenty-three years of service in the university. He was born in 1842.

A. M. G. FLOQUET, professor of mathematical analysis at the University of Nancy has been retired as honorary professor. He was born in 1849.

Professor H. S. WHITE, of Vassar College, spent the summer at the Bureau of Standards, Washington.

Dr. J. W. GLOVER, professor of mathematics and insurance at the University of Michigan, spent the month of September in New York City, serving as Acting President of the Teachers Insurance and Annuity Association of America. Professor Glover is a member of the Board of trustees and acted for Dr. H. S. PRITCHETT, who is spending the summer in the West.

The *Bulletin of the American Mathematical Society* announces that the Royal Institute of Venice has awarded the Querini-Stampalia prize to Professor G. D. BIRKHOFF, of Harvard University, for his papers "The restricted problem of three bodies" (*Rendiconti del Circolo Matematico di Palermo*, volume 39, 1915), and "Dynamical systems with two degrees of Freedom" (*Transactions of the American Mathematical Society*, volume 18, 1917). The prize had been offered for some important advances in the theory of periodic solutions of differential equations. Its value is 3,000 crowns.

During the past summer Assistant Professor LOUISA M. WEBSTER of the department of mathematics of Hunter College conducted a summer high school for girls. She originated this self-supporting school eleven years ago and has served throughout as its director. Instruction was given this summer to nearly fourteen hundred girls. The work is so coordinated with that of the New York City high schools that these have all agreed to accept its ratings. A similar school for boys is projected for next year.

At the University of Minnesota, a committee of twenty members of the Faculty of the Arts College was organized last year, with Professor R. M. BARTON of the mathematics department as chairman, for the purpose of meeting all students in personal conferences. These conferences serve to evince a healthy interest in the scholarship of the pupils, provide for their registration, and, particularly for the freshmen and sophomores, have a distinctly moral bearing. The results of some forty-three hundred conferences during the second semester were found gratifying; the students were appreciative of this personal attention and were through this agency able largely to clarify the notions of the purposes of their university study. This is a new manifestation of the increasing tendency of instructors over the country to meet students outside of the classroom in an advisory capacity, a plan that has been found of great value for departmental purposes by the staff of the department of mathematics in a few institutions.

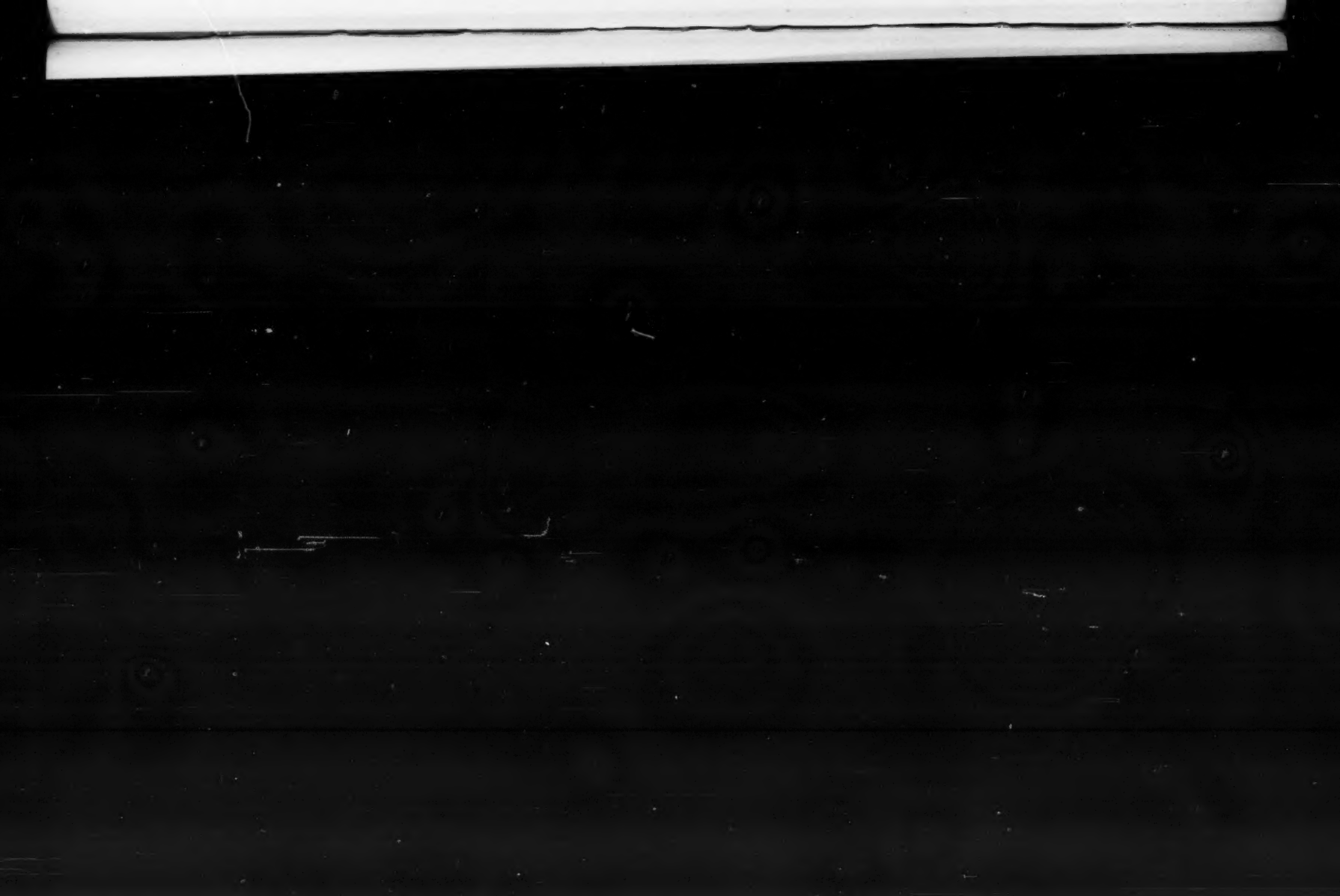
Degrees of Doctor of Philosophy in mathematics were conferred in the academic year 1918–1919 as follows, with theses as indicated:

UNIVERSITY OF CHICAGO: GEORGE HOFFMAN CRESSE: "On the class number of binary quadratic forms."

HARVARD UNIVERSITY: LOUIS BRAND: "I. On linear equations with an infinite number of variables. II. On infinite systems of linear integral equations. III. Flexual deflections and statistically indeterminate beams." CLARENCE NEWTON REYNOLDS: "On the zeros of solutions of linear differential equations." CHAN-CHAN TSOO: "The geometry of a Non-Euclidean line-sphere transformation."

The degree of Doctor of Philosophy in education was conferred, at Columbia University, upon F. C. TOUTON, whose dissertation was on, "The solution of an original exercise in geometry."

The twenty-sixth summer meeting of the American Mathematical Society was held at the University of Michigan, September 2–4, in conjunction with meetings of the American Astronomical Society and the Mathematical Association of America. Eighty-one members were in attendance, the largest number on record. A report of a joint session of the three organizations on Thursday afternoon and of the joint dinner is given elsewhere in this number of the MONTHLY. Thirty-three papers were presented at the other session, titles and abstracts of which are given in the November number of the *Bulletin of the American Mathematical Society*. Professors BEMAN, CURTISS, SNYDER and BLISS presided at the various sessions and Professors KARPINSKI and E. J. MOULTON acted as secretary.



MATHEMATICS AND STATISTICS, WITH AN ELEMENTARY ACCOUNT OF THE CORRELATION COEFFICIENT AND THE CORRELATION RATIO.[1]

By EDWARD V. HUNTINGTON, Harvard University.

1. Introduction.—The first president of the Association, Professor E. R. Hedrick, in his retiring address in 1917, dwelt at length on the important position which this Association should take in relation to the large and growing field of applied mathematics. The Association should accept as perhaps its primary obligation the duty of interpreting the results of pure mathematics to the workers in the field of applied mathematics. This does not mean the "degradation of pure mathematics to utilitarian purposes." It means rather the *search for identity of essential form among apparently diverse problems*. When once the essential form of a problem has been recognized, the marvelously compact and sure analysis of formal mathematics either supplies directly the solution or illuminates the nature of the difficulty. This search for identity of form among the diversities of practical problems is then the task of the interpreter—a task which demands on the one hand a quick sympathy with the needs of the practical sciences, and on the other hand an unswerving loyalty to the rigorous ideals of the purely formal doctrines. The experience of the war has only served to emphasize the ever-growing need of such interpreters and the importance of the work of codification of problems which only they can perform.

2. The Importance of Mathematics in Modern Statistics. "Biometrika."—With this brief word of introduction, I desire to bring to the attention of the Association the opportunities for such interpretative service presented in a comparatively new field of mathematics, namely, the field of mathematical statistics.

Mathematicians as such seem to me to have been slow to enter this field. Of the professional mathematicians in this country only about a dozen have thought it worth while to join the American Statistical Association (one of the oldest learned societies in the United States, founded in 1839 and now having over 800 members). Of the published papers read before the American Mathematical Society during the last five years, only three or four have had any relation to statistics. The very terminology of modern statistical method is unfamiliar to the great majority of professional mathematicians. Few of us, I fancy, have ever heard of the "tetrachoric functions," "homoscedastic linear regressions," or "mesokurtic skew distributions." Most of the development of the science has been left to the economists, the actuaries, the biologists, the psychologists, and, more recently, the pedagogues. The result has been a wide scattering of the

[1] Retiring Address of the President of the Mathematical Association of America, read at the summer meeting, Ann Arbor, Michigan, at a joint session of the Mathematical Association of America, the American Mathematical Society, and the American Astronomical Society, Sept. 4, 1919. Complete references to papers cited only by title will be found in § 14.